

L^1 -determined ideals in group algebras of exponential Lie groups

O. Ungermann

Abstract. A locally compact group G is said to be $*$ -regular if the natural map $\Psi : \text{Prim } C^*(G) \rightarrow \text{Prim}_* L^1(G)$ is a homeomorphism with respect to the Jacobson topologies on the primitive ideal spaces $\text{Prim } C^*(G)$ and $\text{Prim}_* L^1(G)$. In 1980 J. Boidol characterized the $*$ -regular ones among all exponential Lie groups by a purely algebraic condition. In this article we introduce the notion of L^1 -determined ideals in order to discuss the weaker property of primitive $*$ -regularity. We give two sufficient criteria for closed ideals I of $C^*(G)$ to be L^1 -determined. Herefrom we deduce a strategy to prove that a given exponential Lie group is primitive $*$ -regular. The author proved in his thesis that all exponential Lie groups of dimension ≤ 7 have this property. So far no counter-example is known. Here we discuss the example $G = B_5$, the only critical one in dimension ≤ 5 .

2000 Mathematics Subject Classification: 43A20; 22D10, 22D20, 22E27.

1 Introduction

Let \mathcal{A} be Banach $*$ -algebra and $C^*(\mathcal{A})$ its enveloping C^* -algebra in the sense of Dixmier, see Chapter 2.7 of [8]. The C^* -norm on $C^*(\mathcal{A})$ is given by

$$|a|_* = \sup_{\pi \in \hat{\mathcal{A}}} |\pi(a)|$$

for all $a \in \mathcal{A}$ where $\hat{\mathcal{A}}$ is the set of equivalence classes of topologically irreducible $*$ -representations of \mathcal{A} in Hilbert spaces. Let $\text{Prim } C^*(\mathcal{A})$ be the set of primitive ideals in $C^*(\mathcal{A})$, and $\text{Prim}_* \mathcal{A}$ the set of kernels of representations in $\hat{\mathcal{A}}$. For ideals I of $C^*(\mathcal{A})$ we define their hull $h(I) = \{P \in \text{Prim } C^*(\mathcal{A}) : P \supset I\}$ in $\text{Prim } C^*(\mathcal{A})$, and for subsets X of $\text{Prim } C^*(\mathcal{A})$ their kernel $k(X) = \cap \{P : P \in X\}$ in $C^*(\mathcal{A})$. In the sequel all ideals are assumed to be two-sided and closed in the respective norm. Closed ideals I of C^* -algebras are automatically involutive and satisfy $I = k(h(I))$, see Proposition 1.8.2 and Theorem 2.6.1 of [8].

Recall that $\text{Prim } C^*(\mathcal{A})$ is a topological space w. r. t. the Jacobson topology, i.e., $X \subset \text{Prim } C^*(\mathcal{A})$ is closed if and only if there exists an ideal I of $C^*(\mathcal{A})$ such that

$X = h(I)$. Likewise we can state the according definitions of hulls and kernels for \mathcal{A} and we provide $\text{Prim}_* \mathcal{A}$ with the Jacobson topology as well. Let I' denote the preimage of the ideal I under the natural map $\mathcal{A} \longrightarrow C^*(\mathcal{A})$. For simplicity we write $I' = I \cap \mathcal{A}$. The map

$$\Psi : \text{Prim } C^*(\mathcal{A}) \longrightarrow \text{Prim}_* \mathcal{A} \text{ given by } \Psi(P) = P' = P \cap \mathcal{A}$$

is continuous and surjective and evidently satisfies $k(\Psi(X)) = k(X) \cap \mathcal{A}$ and $h(I) \subset \Psi^{-1}(h(I'))$. The next definition is basic for the subsequent investigation.

Definition 1.1. A closed ideal I of $C^*(\mathcal{A})$ is called \mathcal{A} -determined if and only if the following (obviously) equivalent conditions hold:

- (i) $I' \subset J'$ implies $I \subset J$ for all ideals J of $C^*(\mathcal{A})$,
- (ii) $I' \subset P'$ implies $I \subset P$ for all $P \in \text{Prim } C^*(\mathcal{A})$, i.e., $h(I) = \Psi^{-1}(h(I'))$,
- (iii) I' is dense in I w. r. t. the C^* -norm,
- (iv) $C^*(\mathcal{A}/I') \cong C^*(\mathcal{A})/I$.

In the introduction of [2] Boidol defined $*$ -regularity of Banach $*$ -algebras. We restate his definition and add the notion of primitive $*$ -regularity.

Definition 1.2. A Banach $*$ -algebra \mathcal{A} is called (primitive) $*$ -regular if and only if every closed (primitive) ideal of $C^*(\mathcal{A})$ is \mathcal{A} -determined.

The group algebra $L^1(G)$ of a locally compact group G is a $*$ -semisimple Banach $*$ -algebra with bounded approximate identities. We say that G is (primitive) $*$ -regular if $L^1(G)$ has this property. Similarly $*$ -regularity of (real) Lie algebras \mathfrak{g} is defined by means of the (unique) connected, simply connected Lie group G with $\text{Lie}(G) = \mathfrak{g}$.

Part (ii) of the next lemma shows that Definition 1.2 is equivalent to Boidol's original definition, a characterization which has already been proved in [4].

Lemma 1.3.

- (i) If \mathcal{A} is primitive $*$ -regular, then $\Psi : \text{Prim } C^*(\mathcal{A}) \longrightarrow \text{Prim}_* \mathcal{A}$ is injective.
- (ii) A Banach $*$ -algebra \mathcal{A} is $*$ -regular if and only if Ψ is a homeomorphism with respect to the Jacobson topologies on $\text{Prim } C^*(\mathcal{A})$ and $\text{Prim}_* \mathcal{A}$.

Proof. If \mathcal{A} is primitive $*$ -regular, then $P = \overline{\Psi(P)}$ is uniquely determined by $\Psi(P)$ for all $P \in \text{Prim } C^*(\mathcal{A})$. This proves (i). In order to prove (ii), let us suppose that \mathcal{A} is $*$ -regular. Since Ψ is a continuous bijection, it suffices to prove that Ψ maps closed sets onto closed sets. But if X is a closed subset of $\text{Prim } C^*(\mathcal{A})$, then there exists a closed ideal I of $C^*(\mathcal{A})$ such that $X = h(I)$ and we see that $\Psi(X) = h(I')$ is closed in $\text{Prim}_* \mathcal{A}$ because I is \mathcal{A} -determined. Now we prove the opposite implication. Assume

that Ψ is a homeomorphism, I a closed ideal of $C^*(\mathcal{A})$, and $P \in \text{Prim } C^*(\mathcal{A})$ such that $I' \subset P'$. Define $X = h(I)$. Since $I' = k(\Psi(X))$, it follows

$$h(I') = h(k(\Psi(X))) = \overline{\Psi(X)} = \Psi(X)$$

because Ψ maps closed sets onto closed sets. Now $P' \in \Psi(X)$ implies $P \in X$ so that $P \supset I$ because Ψ is injective. This proves the asserted equivalence. \square

Because of its technical importance we state the following fact as a lemma, but we omit the easy proof.

Lemma 1.4. *Let $I \subset J$ be closed ideals of $C^*(\mathcal{A})$ such that I is \mathcal{A} -determined. Then J is \mathcal{A} -determined if and only if the ideal J/I of $C^*(\mathcal{A})/I = C^*(\mathcal{A}/I')$ is \mathcal{A}/I' -determined.*

This lemma can be applied in the following situation: If A is a closed normal subgroup of G and $\dot{G} = G/A$, then $T_A f(\dot{x}) = \int_A f(xa) da$ defines a quotient map of Banach $*$ -algebras from $L^1(G)$ onto $L^1(\dot{G})$ which extends to a quotient map from $C^*(G)$ onto $C^*(\dot{G})$, compare p. 68 of [29]. It is easy to see that $I = \ker_{C^*(G)} T_A$ is $L^1(G)$ -determined.

Lemma 1.5. *A finite intersection of \mathcal{A} -determined ideals is \mathcal{A} -determined.*

Proof. Let I_1 and I_2 be \mathcal{A} -determined ideals of $C^*(\mathcal{A})$. Let $P \in \text{Prim } C^*(\mathcal{A})$ such that $I'_1 \cdot I'_2 \subset I'_1 \cap I'_2 \subset P'$. Since P' is a prime ideal of \mathcal{A} , it follows $I'_1 \subset P'$ or $I'_2 \subset P'$. Since I_1 and I_2 are \mathcal{A} -determined, we obtain $I_1 \subset P$ or $I_2 \subset P$ and thus $I_1 \cap I_2 \subset P$. Consequently $I_1 \cap I_2$ is \mathcal{A} -determined and the assertion of this lemma follows by induction. \square

Remark 1.6. Here are a few examples of $*$ -regular Banach $*$ -algebras: If G is a connected locally compact group such that its Haar measure has polynomial growth, then G is $*$ -regular. Boidol proved this fact in Theorem 2 of [4] based on ideas of Dixmier in [7]. Jenkins has shown in Theorem 1.4 of [15] that connected nilpotent Lie groups have polynomial growth. If G is a metabelian connected locally compact group, then G is $*$ -regular, see Theorem 3.5 of [2]. Moreover the following is true: If G is a compactly generated, locally compact group with polynomial growth and if w is a symmetric weight function on G which satisfies the non-abelian-Beurling-Domar condition (BDna) of [10], then $L^1(G, w)$ is $*$ -regular. Compare Proposition 5.2 and Theorem 5.8 of [10].

In the next paragraphs we formulate sufficient criteria for ideals of the group algebra $C^*(G)$ of exponential Lie groups to be $L^1(G)$ -determined, see Proposition 2.12 and Proposition 4.14.

2 Inducing primitive ideals from a stabilizer

We shall use the concept of the adjoint algebra (double centralizer algebra) of a Banach $*$ -algebra, compare Paragraph 3 of [16] and Chapter 2.3 of [28]. Let $\mathcal{C}_0(G)$ denote the

continuous functions of compact support on G . If H is a closed subgroup of G , then $\mathcal{C}_0(H)$ acts as an algebra of double centralizers on $\mathcal{C}_0(G)$ by convolution

$$(a * f)(x) = \int_H a(h) f(h^{-1}x) dh$$

from the left, and by

$$(f * a)(x) = \int_H f(xh) \Delta_{G,H}(h^{-1}) a(h^{-1}) dh$$

from the right where $\Delta_{G,H}(h) = \Delta_H(h)\Delta_G(h)^{-1}$. These actions extend to actions of $C^*(H)$ on $C^*(G)$ such that $(a * f)^* = f^* * a^*$ and $f * (a * g) = (f * a) * g$ for all $f, g \in C^*(G)$ and $a \in C^*(H)$.

Definition 2.1. Let H be a closed subgroup of the locally compact group G . If J is an ideal of $C^*(H)$, then

$$\text{ind}_H^G(J) = (C^*(G) * J * C^*(G))^-$$

denotes the induced ideal of $C^*(G)$. If I is an ideal of $C^*(G)$, then the ideal

$$\text{res}_H^G(I) = \{a \in C^*(H) : a * C^*(G) \subset I\}$$

is its restriction to H . An ideal I of $C^*(G)$ is said to be induced from H if there exists an ideal J of $C^*(H)$ such that $I = \text{ind}_H^G(J)$.

If $I = \ker_{C^*(G)} \pi$ for some unitary representation π of G , then $\text{res}_H^G(I) = \ker_{C^*(H)} \pi$. If I is induced from H , then $I = \text{ind}_H^G(\text{res}_H^G(I))$. Note that $I = \text{ind}_H^G(J)$ is minimal among all ideals of $C^*(G)$ whose restriction contains J .

It is interesting to compare our definition of induced ideals to that of Green and Rieffel in Section 3 of [13] involving C^* -imprimitivity bimodules. To this end we assume that there exists a G -invariant measure on the homogeneous space G/H so that the character $\Delta_{G,H}$ of H is trivial. This is the case e.g. if H is a normal subgroup of G . We follow the considerations of Section 4 of Rieffel's article [30]. Note that the right action of $\mathcal{C}_0(H)$ on $X_0 = \mathcal{C}_0(G)$ defined in [30] coincides with that of convolution from the right because the function $\gamma = \Delta_{G,H}^{-1/2}$ used there is also trivial. The $\mathcal{C}_0(H)$ -valued inner product

$$\langle f | g \rangle_{\mathcal{C}_0(H)}(h) = (f^* * g)(h) = \int_G \overline{f(y)} g(yh) dy$$

defines a norm $\|f\|_{C^*(H)} = |\langle f | f \rangle_{\mathcal{C}_0(H)}|^{1/2}$ on X_0 where the norm on the right is the C^* -norm of $C^*(H)$. Further $\mathcal{C}_0(G)$ acts on X_0 by convolution from the left so that

X_0 becomes a $\mathcal{C}_0(G)$ - $\mathcal{C}_0(H)$ -bimodule, and $\langle f | g \rangle_{\mathcal{C}_0(G)} = f * g^*$ defines a $\mathcal{C}_0(G)$ -valued inner product ${}_{C^*(G)}\langle \cdot | \cdot \rangle$ on X_0 . Completion of X_0 with respect to the norm $|\cdot|_{C^*(H)}$ gives a right- $C^*(H)$ -rigged space X on which $C^*(G)$ acts from the left. From

$$\text{ind}_H^G(J) = \overline{\text{span}} \{ {}_{C^*(G)}\langle f * a | g \rangle : f, g \in X \text{ and } a \in J \} = X - \text{ind}_H^G(J)$$

we learn that, at least in the case of $\Delta_{G,H}$ being trivial, our definition coincides with that of Rieffel and Green.

It is well-known that for C^* -**imprimitivity** bimodules X the Rieffel correspondence $X - \text{ind}_H^G$ is compatible with inducing representations in the sense that

$$(2.2) \quad X - \text{ind}_H^G(\ker \sigma) = \ker(X - \text{ind}_H^G \sigma),$$

compare Chapter 3.3 of [28]. But in general the bimodule X from above is **not** a $C^*(G)$ - $C^*(H)$ -imprimitivity bimodule because the crucial equality ${}_{C^*(G)}\langle f | g \rangle * h = f * \langle g | h \rangle_{C^*(H)}$ is not necessarily satisfied. The norms $|\cdot|_{C^*(H)}$ and $|\cdot|_{C^*(G)}$ might be different. In fact, the imprimitivity algebra of the $C^*(H)$ -rigged space X is known to be isomorphic to the covariance algebra $C^*(G, \mathcal{C}_\infty(G/H))$. As we will see, Equation 2.2 holds true for the C^* -bimodule X defined above if G/H is amenable.

In analogy to results of Leptin [17] and Hauenschild, Ludwig [14] for the L^1 -case, we will characterize those ideals I of $C^*(G)$ which are induced from a given closed normal subgroup H of G . This turns out to be possible if H is normal and G/H amenable. In order to prepare the proof of Theorem 2.6 we recall the well-known restriction-induction-lemma of Fell, see Theorem 3.1 and Lemma 4.2 of [11]. A proof can also be found on p. 32 of [19]. We presume the definition of induced representations.

Lemma 2.3. *Let H be a closed subgroup of a locally compact group G . Let π be a unitary representation of G and $\pi|_H$ its restriction to H .*

- (i) *If τ is a unitary representation of H , then the Kronecker product $\text{ind}_H^G((\pi|_H) \otimes \tau)$ is unitarily equivalent to $\pi \otimes \text{ind}_H^G \tau$.*
- (ii) *In particular $\text{ind}_H^G(\pi|_H)$ is unitarily equivalent to $\pi \otimes \lambda$ where λ denotes the left regular representation of G in $L^2(G/H)$.*

Note that conjugation $f^z(x) = \Delta_G(z^{-1}) f(xz^{-1})$ for $f \in L^1(G)$ and $z \in G$ extends to a strongly continuous action of G on $C^*(G)$ by isometric automorphisms. Using an approximate identity of $C^*(G)$, one can prove that every closed ideal I of $C^*(G)$ is two-sided translation-invariant, and hence invariant under conjugation, i.e. $I^z = I$.

If H is a closed normal subgroup of G , then G acts on H by conjugation $n^z = z^{-1}nz$. Further $a^z(n) = \delta(z^{-1}) a(nz^{-1})$ for $a \in L^1(H)$ and $z \in G$ yields a strongly continuous, isometric action of G on $C^*(H)$. If I is a closed ideal of $C^*(G)$, then $J = \text{res}_H^G(I)$ is a G -invariant ideal of $C^*(H)$, i.e. $J^z = J$, because $(a * f)^z = a^z * f^z$ and $I^z = I$.

Lemma 2.4. *Let H be a closed normal subgroup of a locally compact group G . Let σ be a unitary representation of H and $\pi = \text{ind}_H^G \sigma$. Then $\pi|_H$ is weakly equivalent to the orbit $G \cdot \sigma$ which means*

$$\ker_{C^*(H)} \pi = k(G \cdot \sigma) = \bigcap_{x \in G} \ker_{C^*(H)} x \cdot \sigma.$$

Proof. Let \mathfrak{H} be the representation space of σ . As usual $\mathcal{C}_0^\sigma(G, \mathfrak{H})$ denotes the vector space of all continuous functions on G which satisfy $\varphi(xh) = \sigma(h)^* \varphi(x)$ for $h \in H$, $x \in G$ and have compact support modulo H . Then $\pi = \text{ind}_H^G \sigma$ is defined in $L_\sigma^2(G, \mathfrak{H})$, the completion of $\mathcal{C}_0^\sigma(G, \mathfrak{H})$ with respect to the L^2 -norm given by integration with respect to the Haar measure of the group G/H . We get

$$\pi(h)\varphi(x) = \varphi(h^{-1}x) = \sigma(h^x) \cdot \varphi(x)$$

for $h \in H$. It follows that $\pi|_H$ is given by $\pi(a)\varphi(x) = \sigma(a^x) \cdot \varphi(x)$ for $a \in C^*(H)$. Hence π is essentially a direct integral of the representations $\{x \cdot \sigma : x \in G\}$ so that the assertion of this lemma becomes clear. \square

The importance of the left regular representation λ of G in $L^2(G/H)$ has already been indicated by Lemma 2.3.

Definition 2.5. Let H be a closed normal subgroup of a locally compact group G . An ideal I of $C^*(G)$ is said to be $(G/H)^\wedge$ -invariant if π is weakly equivalent to $\pi \otimes \lambda$ (in symbols $\pi \approx \pi \otimes \lambda$) for all unitary representations π of G such that $I = \ker_{C^*(G)} \pi$.

Theorem 1 of [12] shows that $\pi \approx \pi \otimes \lambda$ for at least one such π is sufficient for I to be $(G/H)^\wedge$ -invariant. Now we can state the announced characterization of induced ideals.

Theorem 2.6. *Let H be a closed normal subgroup of a locally compact group G such that G/H is amenable. Then there are equivalent:*

- (i) $I = \text{ind}_H^G(\text{res}_H^G(I))$ is induced from H .
- (ii) $I = \ker_{C^*(G)} \pi$ is the kernel of some induced representation $\pi = \text{ind}_H^G \sigma$.
- (iii) I is $(G/H)^\wedge$ -invariant.

Proof. First we verify (i) \Rightarrow (ii). Suppose that I is induced from H . Since $J = \text{res}_H^G(I)$ is a G -invariant ideal of $C^*(H)$, its hull $\Omega = h(J) \subset \widehat{H}$ is G -invariant, too. Define $\sigma = \sum_{\tau \in \Omega}^\oplus \tau$ and $\pi = \text{ind}_H^G \sigma$. Lemma 2.4 implies $\ker_{C^*(H)} \pi = k(G \cdot \sigma) = k(\Omega) = J$. Hence $I = \text{ind}_H^G(J) \subset \ker_{C^*(G)} \pi$. We must prove the opposite inclusion: Let $\rho \in \widehat{G}$ be arbitrary such that $I \subset \ker_{C^*(G)} \rho$. Then $k(G \cdot \sigma) = J \subset \ker_{C^*(H)} \rho$ which means that $\rho|_H$ is weakly contained in $G \cdot \sigma$ (in symbols $\rho|_H \ll G \cdot \sigma$). Since G/H is amenable, we have $1_G \ll \lambda = \text{ind}_H^G 1_H$ and hence $\rho \otimes 1_G \ll \rho \otimes \lambda$ by Theorem 1 of [12]. For inducing

representations is continuous w. r. t. the Fell topologies of \widehat{H} and \widehat{G} , it follows from part (ii) of Lemma 2.3 that

$$\rho \cong \rho \otimes 1_G \ll \rho \otimes \lambda \cong \text{ind}_H^G(\rho|H) \ll \text{ind}_H^G(G \cdot \sigma) \approx \text{ind}_H^G \sigma = \pi$$

because the representations $\text{ind}_H^G(z \cdot \sigma)$, $z \in G$, are all unitarily equivalent. Thus $\ker_{C^*(G)} \pi \subset \ker_{C^*(G)} \rho$. Since I is the intersection of all primitive ideals of $C^*(G)$ containing I by Theorem 2.9.7 of [8], we obtain $I = \ker_{C^*(G)} \pi$.

Next we show (ii) \Rightarrow (iii). Suppose that $I = \ker_{C^*(G)} \pi$ for some $\pi = \text{ind}_H^G \sigma$. By Lemma 2.4 we know $\pi|H \approx G \cdot \sigma$. Thus $\pi \otimes \lambda \cong \text{ind}_H^G(\pi|H) \approx \text{ind}_H^G \sigma = \pi$ which proves I to be $(G/H)^\wedge$ -invariant.

Finally we prove (iii) \Rightarrow (i). Suppose that I is $(G/H)^\wedge$ -invariant. Set $J = \text{res}_H^G(I)$. Clearly $\text{ind}_H^G(J) \subset I$. It remains to verify the opposite inclusion: Choose a unitary representation π of G such that $I = \ker_{C^*(G)} \pi$. Let $\rho \in \widehat{G}$ be arbitrary such that $\text{ind}_H^G(J) \subset \ker_{C^*(G)} \rho$. Then $\ker_{C^*(H)} \pi = J \subset \ker_{C^*(H)} \rho$ and hence $\rho|H \ll \pi|H$. Since G/H is amenable, we get

$$\rho = \rho \otimes 1 \ll \rho \otimes \lambda = \text{ind}_H^G(\rho|H) \ll \text{ind}_H^G(\pi|H) = \pi \otimes \lambda \approx \pi$$

because I is $(G/H)^\wedge$ -invariant. Thus $I = \ker_{C^*(G)} \pi \subset \ker_{C^*(G)} \rho$. Now Theorem 2.9.7 of [8] implies $I = \text{ind}_H^G(J)$. The proof is complete. \square

The proof of (i) \Rightarrow (ii) of Theorem 2.6 shows that $J = \text{res}_H^G(\text{ind}_H^G(J))$ for every G -invariant ideal J of $C^*(H)$. The preceding results can be summarized as follows:

Theorem 2.7. *Let H be a closed normal subgroup of a locally compact group G such that G/H is amenable. Induction and restriction give bijections between the set of all $(G/H)^\wedge$ -invariant ideals I of $C^*(G)$ and the set of all G -invariant ideals J of $C^*(H)$ which are inverses of one another.*

An immediate consequence is

Corollary 2.8. *Let H be a closed normal subgroup of a locally compact group G such that G/H is amenable. If the ideals $\{I_k : k \in \Lambda\}$ of $C^*(G)$ are induced from H , then their intersection $I = \bigcap \{I_k : k \in \Lambda\}$ is also induced from H .*

Proof. Let π_k be a unitary representation of G such that $I_k = \ker_{C^*(G)} \pi_k$. We know $\pi_k \otimes \lambda \approx \pi_k$ by Theorem 2.6. If we define $\pi = \sum_{k \in \Lambda}^{\oplus} \pi_k$, then $I = \ker_{C^*(G)} \pi$ and $\pi \otimes \lambda \approx \{\pi_k \otimes \lambda : k \in \Lambda\} \approx \{\pi_k : k \in \Lambda\} \approx \pi$. Thus I induced from H by Theorem 2.6. \square

Suppose that H is a coabelian normal subgroup of G so that G/H is amenable as an abelian group. In this case $(\chi \cdot f)(x) = \chi(x)f(x)$ for $f \in L^1(G)$ extends to an isometric, strongly continuous action of the Pontryagin dual $(G/H)^\wedge$ on $C^*(G)$. Note that $\pi(\chi \cdot f) = (\pi \otimes \chi)(f)$ for any unitary representation π of G .

Corollary 2.9. *Let H be a coabelian normal subgroup of G . An ideal I of $C^*(G)$ is induced from H if and only if it is $(G/H)^\wedge$ -invariant in the sense that $\chi \cdot I = I$ for all $\chi \in (G/H)^\wedge$.*

Proof. Let π be a unitary representation of G such that $I = \ker_{C^*(G)} \pi$. Theorem 2.6 shows that I is induced from H if and only if $\pi \approx \pi \otimes \lambda$. Since G/H is abelian, it follows $\lambda \approx \{\chi : \chi \in (G/H)^\wedge\}$ and hence

$$\ker_{C^*(G)} \pi \otimes \lambda = \bigcap_{\chi \in (G/H)^\wedge} \ker_{C^*(G)} \pi \otimes \chi \subset \ker_{C^*(G)} \pi.$$

Thus we see that $\pi \otimes \lambda \approx \pi$ if and only if $\ker_{C^*(G)} \pi \otimes \chi = \ker_{C^*(G)} \pi$ for all χ . This is the case if and only if $\chi \cdot I = I$ for all $\chi \in (G/H)^\wedge$. \square

The preceding corollary displays a close connection to the L^1 -results of Leptin and Hauenschild, Ludwig. In 1968 Leptin characterized the induced ideals of generalized L^1 -algebras / twisted covariance algebras $L^1(G, \mathcal{A}, \tau)$, see Satz 8 and Satz 9 of [17]. His results imply Theorem 2.10 and Lemma 2.11 below. In 1981 Hauenschild and Ludwig gave a different proof of Theorem 2.10 using L^1 - L^∞ -duality, see Theorem 2.3 of [14]. These L^1 -results hold true without the additional assumption of amenability.

An ideal I of $L^1(G)$ is said to be induced from H if there exists an ideal J of $L^1(H)$ such that $I = \text{ind}_H^G(J) = (L^1(G) * J * L^1(G))^-$.

Theorem 2.10. *Let H be a closed normal subgroup of a locally compact group G . An ideal I of $L^1(G)$ is induced from H if and only if it is $\mathcal{C}_\infty(G/H)$ -invariant. Induction and restriction gives a bijection between the set of all $\mathcal{C}_\infty(G/H)$ -invariant ideals I of $L^1(G)$ and the set of all G -invariant ideals J of $L^1(H)$.*

Here $\mathcal{C}_\infty(G/H)$ denotes the continuous functions on G/H vanishing at infinity.

Lemma 2.11. *Let H be a closed normal subgroup of a locally compact group G . If J is a closed, G -invariant ideal of $L^1(H)$, then $J * L^1(G)$ is contained in the closure of $L^1(G) * J$. Similarly $L^1(G) * J$ is contained in the closure of $J * L^1(G)$.*

This implies $I = \text{ind}_H^G(J) = (J * L^1(G))^- = (L^1(G) * J)^-$. For G -invariant C^* -ideals we even know $J * C^*(G) = C^*(G) * J$ by Corollary 2.3 of the main lemma in [31].

Now we can state our first criterion for ideals of $C^*(G)$ to be $L^1(G)$ -determined.

Proposition 2.12. *Let G be a locally compact group and H a $*$ -regular closed subgroup. If the ideal I of $C^*(G)$ is induced from H , then I is $L^1(G)$ -determined.*

Proof. Let $J = \text{res}_H^G(I)$ so that $I = \text{ind}_H^G(J)$. If $\rho \in \widehat{G}$ with $I' = I \cap L^1(G) \subset \ker_{L^1(G)} \rho$, then $J' \subset \ker_{L^1(H)} \rho$. Since H is $*$ -regular, it follows $J \subset \ker_{C^*(H)} \rho$. This implies $I = \text{ind}_H^G(J) \subset \text{ind}_H^G(\ker_{C^*(H)} \rho) \subset \ker_{C^*(G)} \rho$. \square

In the rest of this article we will focus on exponential Lie groups (i.e. connected, simply connected, solvable Lie groups G such that the exponential map $\exp : \mathfrak{g} \rightarrow G$ is a global diffeomorphism). We will use the construction of irreducible representations $\pi = \mathcal{K}(f) = \text{ind}_P^G \chi_f$ via Pukanszky / Vergne polarizations \mathfrak{p} at f and the bijectivity of the Kirillov map $\mathcal{K} : \mathfrak{g}^* / \text{Ad}^*(G) \rightarrow \widehat{G}$, see Chapters 4 and 6 of [1], and Chapter 1 of [19]. Mostly we regard \mathcal{K} as a map from \mathfrak{g}^* onto \widehat{G} which is constant on coadjoint orbits.

Lemma 2.13. *Let G be an exponential Lie group with Lie algebra \mathfrak{g} . Let $f \in \mathfrak{g}^*$ and $q \in [\mathfrak{g}, \mathfrak{g}]^\perp \subset \mathfrak{g}^*$. If we define $\pi = \mathcal{K}(f)$ and the character $\alpha(\exp X) = e^{iq(X)}$ of G , then $\mathcal{K}(f + q)$ and $\pi \otimes \alpha$ are unitarily equivalent.*

Proof. Let $\mathfrak{p} \subset \mathfrak{g}$ be a Pukanszky polarization at f , and hence also at $f + q$. Let χ_f and χ_{f+q} denote characters of P with differential f and $f + q$. By definition of the Kirillov map we have $\pi = \text{ind}_P^G \chi_f$ and $\rho = \mathcal{K}(f + q) = \text{ind}_P^G \chi_{f+q}$. Now one verifies easily that $(U\varphi)(x) = \overline{\alpha(x)} \varphi(x)$ defines a unitary isomorphism from $\mathfrak{H}_\pi = L^2_{\chi_f}(G)$ onto $\mathfrak{H}_\rho = L^2_{\chi_{f+q}}(G)$ such that $\rho = U(\pi \otimes \alpha)U^{-1}$. This proves our claim. \square

The next proposition enlightens the significance of the 'stabilizer' M .

Proposition 2.14. *Let G be an exponential Lie group, \mathfrak{n} a coabelian (nilpotent) ideal of its Lie algebra \mathfrak{g} , and $f \in \mathfrak{g}^*$. Let M denote the connected subgroup of G with Lie algebra $\mathfrak{m} = \mathfrak{g}_f + \mathfrak{n}$. If $\pi = \mathcal{K}(f)$, then the primitive ideal $\ker_{C^*(G)} \pi$ is induced from the stabilizer M .*

Proof. First we observe that the orbit $\text{Ad}^*(G)f$ is saturated over \mathfrak{m} : Let G_l denote the (connected) stabilizer of $l = f|_{\mathfrak{n}}$ in G . Since $\text{Ad}^*(G_l)f = f + \mathfrak{m}^\perp$, it follows $\text{Ad}^*(G)f = \text{Ad}^*(G)f + \mathfrak{m}^\perp$, compare p. 23 of [1]. Now the preceding lemma implies $\pi \otimes \alpha = \mathcal{K}(f + q) = \mathcal{K}(f) = \pi$ for all $q \in \mathfrak{m}^\perp$ and characters $\alpha(\exp X) = e^{iq(X)}$ of G/M proving $\ker_{C^*(G)} \pi$ to be $(G/M)^\wedge$ -invariant. Hence $\ker_{C^*(G)} \pi$ is induced from M by Corollary 2.9. \square

3 The ideal theory of $*$ -regular exponential Lie groups

The results of this subsection are not new. They can be found in Boidol's paper [2], and in a more general context in [3]. For the convenience of the reader we give a short proof for the if-part of Theorem 5.4 of [2] using the results of the previous section. The following definition has been adapted from the introduction of [3].

Definition 3.1. Let G be a locally compact group. If A is a closed normal subgroup of G and $\dot{G} = G/A$, then T_A denotes the quotient map from $C^*(G)$ onto $C^*(\dot{G})$. We say that a closed ideal I of $C^*(G)$ is essentially induced from a $*$ -regular subgroup if there exist closed subgroups $A \subset H$ of G with A normal in G such that the following conditions are satisfied:

$$(i) \quad \ker_{C^*(G)} T_A \subset I,$$

- (ii) H/A is $*$ -regular,
- (iii) I is induced from H in the sense of Definition 2.1.

Recall that connected locally compact groups whose Haar measure have polynomial growth are $*$ -regular, and that connected nilpotent Lie groups have polynomial growth. If we pass to the quotient \dot{G} by Proposition 1.4, then it follows from Proposition 2.12 that all ideals I of $C^*(G)$ which are essentially induced from a $*$ -regular subgroup are $L^1(G)$ -determined.

Definition 3.2. Let \mathfrak{g} be an exponential Lie algebra and $\mathfrak{n} = [\mathfrak{g}, \mathfrak{g}]$ its commutator ideal. We say that \mathfrak{g} satisfies condition (R) if the following is true: If $f \in \mathfrak{g}^*$ is arbitrary and $\mathfrak{m} = \mathfrak{g}_f + \mathfrak{n}$ is its stabilizer, then $f = 0$ on $\mathfrak{m}^\infty = \bigcap_{k=1}^\infty C^k \mathfrak{m}$. Here the $C^k \mathfrak{m}$ are the ideals of the descending central series. Recall that \mathfrak{m}^∞ is the smallest ideal of \mathfrak{m} such that $\mathfrak{m}/\mathfrak{m}^\infty$ is nilpotent.

Note that the stabilizer $\mathfrak{m} = \mathfrak{g}_f + \mathfrak{n}$ depends only on the orbit $\text{Ad}^*(G)f$. The following observation is extremely useful: Let $f \in \mathfrak{g}^*$ and $\mathfrak{m} = \mathfrak{g}_f + \mathfrak{n}$ be its stabilizer such that $\mathfrak{m}/\mathfrak{m}^\infty$ is nilpotent. If $\gamma_1, \dots, \gamma_r$ are the roots of \mathfrak{g} , then we define the ideal $\tilde{\mathfrak{m}} = \bigcap_{i \in S} \ker \gamma_i$ of \mathfrak{g} where $S = \{i : \ker \gamma_i \supset \mathfrak{m}\}$. It is easy to see that $\mathfrak{m} \subset \tilde{\mathfrak{m}}$ and that $\tilde{\mathfrak{m}}/\mathfrak{m}^\infty$ is nilpotent, too. Further there are only finitely many ideals $\tilde{\mathfrak{m}}$ of this kind.

Theorem 3.3. *Let G be an exponential Lie group such that its Lie algebra \mathfrak{g} satisfies condition (R). Then any ideal I of $C^*(G)$ is a finite intersection of ideals which are essentially induced from a nilpotent normal subgroup. In particular G is $*$ -regular.*

Proof. Let $I \triangleleft C^*(G)$ be arbitrary. Since $I = k(h(I))$ by Theorem 2.9.7 of [8], there is a closed, $\text{Ad}^*(G)$ -invariant subset Λ of \mathfrak{g}^* such that $I = \bigcap \{\ker_{C^*(G)} \mathcal{K}(f) : f \in \Lambda\}$. Further there exists a decomposition $\Lambda = \bigcup_{k=1}^r \Lambda_k$ and ideals $\{\tilde{\mathfrak{m}}_k : 1 \leq k \leq r\}$ of \mathfrak{g} as in the preceding remark such that $\mathfrak{g}_f + \mathfrak{n} \subset \tilde{\mathfrak{m}}_k$ for all $f \in \Lambda_k$ where $\mathfrak{n} = [\mathfrak{g}, \mathfrak{g}]$. By induction in stages it follows from Proposition 2.14 that $\ker_{C^*(G)} \mathcal{K}(f)$ is induced from \tilde{M}_k for all $f \in \Lambda_k$. Let us define $I_k = \bigcap \{\ker_{C^*(G)} \mathcal{K}(f) : f \in \Lambda_k\}$. Since $f = 0$ on $\tilde{\mathfrak{m}}_k^\infty$ by condition (R) and $\tilde{M}_k/\tilde{M}_k^\infty$ is nilpotent, we conclude from Corollary 2.8 that I_k is essentially induced from a nilpotent (and hence $*$ -regular) normal subgroup. Finally Lemma 1.5 implies that the ideal $I = \bigcap_{k=1}^r I_k$ is $L^1(G)$ -determined. \square

4 Closed orbits in the unitary dual of the nilradical

First we recall how to compute the C^* -kernel of $\pi|N$ in the Kirillov picture, compare Theorem 9 in Section 5 of Chapter 1 in [19]. Note that the linear projection $r : \mathfrak{g}^* \twoheadrightarrow \mathfrak{n}^*$ given by restriction is $\text{Ad}^*(G)$ -equivariant so that $r(\text{Ad}^*(G)f) = \text{Ad}^*(G)l$.

Lemma 4.1. *Let G be an exponential Lie group and \mathfrak{n} a coabelian ideal of its Lie algebra \mathfrak{g} . Let $f \in \mathfrak{g}^*$, $\pi = \mathcal{K}(f) \in \hat{G}$, $l = f|_{\mathfrak{n}}$, and $\sigma = \mathcal{K}(l) \in \hat{N}$. Then*

$$(4.2) \quad \ker_{C^*(N)} \pi = k(G \cdot \sigma) = \bigcap_{h \in \text{Ad}^*(G)l} \ker_{C^*(N)} \mathcal{K}(h) .$$

Proof. The second equality is obvious because the Kirillov map of N is G -equivariant. By induction it suffices to prove the first equality in the case $\dim \mathfrak{g}/\mathfrak{n} = 1$. First we assume $\mathfrak{g}_f \subset \mathfrak{n}$. Let us choose a Pukanszky polarization $\mathfrak{p} \subset \mathfrak{n}$ at $l \in \mathfrak{n}^*$. It is easy to see that $\mathfrak{p} \subset \mathfrak{g}$ is also a Pukanszky polarization at $f \in \mathfrak{g}^*$. By induction in stages we obtain $\pi = \text{ind}_P^G \chi_f = \text{ind}_N^G \sigma$ so that $\ker_{C^*(G)} \pi = k(G \cdot \sigma)$ by Lemma 4.1. Next we assume $\mathfrak{g}_f \not\subset \mathfrak{n}$. Using the concept of Vergne polarizations passing through \mathfrak{n} we see that there exists a Pukanszky polarization $\mathfrak{p} \subset \mathfrak{g}$ at $f \in \mathfrak{g}^*$ such that $\mathfrak{q} = \mathfrak{p} \cap \mathfrak{n}$ is a Pukanszky polarization at $l \in \mathfrak{n}^*$. We point out that the restriction of functions from G to N gives a linear isomorphism $\mathcal{C}_0^{\chi_f}(G) \rightarrow \mathcal{C}_0^{\chi_l}(N)$ which extends to a unitary isomorphism U from $\mathfrak{H}_\pi = L_{\chi_f}^2(G)$ onto $\mathfrak{H}_\sigma = L_{\chi_l}^2(N)$. Clearly U intertwines $\pi|_H$ and σ . On the other hand $\mathfrak{g} = \mathfrak{g}_f + \mathfrak{n}$ implies $\text{Ad}^*(G)l = \text{Ad}^*(N)l$ and thus $G \cdot \sigma = \{\sigma\}$. This proves $\ker_{C^*(N)} \pi = \ker_{C^*(N)} \sigma = k(G \cdot \sigma)$. \square

In the sequel we suppose that \mathfrak{n} is nilpotent and coabelian. Note that the orbit $G \cdot \sigma \subset \widehat{N}$ is uniquely determined by Equality (4.2) because it is locally closed (open in its closure): Pukanszky showed in Corollary 1 of [27] that $\text{Ad}^*(G)l$ is locally closed in \mathfrak{n}^* and Brown proved in [5] that the Kirillov map of the connected, simply connected, nilpotent Lie group N is a homeomorphism.

Our main result is Proposition 4.14 which states that the primitive ideal $\ker_{C^*(G)} \pi$ is $L^1(G)$ -determined if $G \cdot \sigma$ is closed in \widehat{N} . This result is a consequence of arguments closely related to the classification of simple $L^1(G)$ -modules, G an exponential Lie group, established by Poguntke in [26].

Let π, f, σ, l be as in Lemma 4.1. It is easy to see that $\mathfrak{g} = \mathfrak{g}_l + \mathfrak{n}$ is sufficient for $G \cdot \sigma$ to be closed in \widehat{N} : Theorem 3.1.4 of [6] implies that $\text{Ad}^*(G)l = \text{Ad}^*(N)l$ is closed in \mathfrak{n}^* because N acts unipotently on \mathfrak{n}^* . Since the Kirillov map of N is a homeomorphism, it follows that $G \cdot \sigma = \{\sigma\}$ is closed in \widehat{N} . Alternatively one can resort to the results of Moore and Rosenberg: It follows from Theorem 1 of [22] that \widehat{N} is a T_1 -space so that its one-point subsets are closed. Let us give a third proof of this fact: Since $L^1(N)$ is symmetric for nilpotent connected Lie groups N by Satz 2 of [23], it follows $\text{Prim}_* L^1(N) = \text{Max } L^1(N)$ by (6) of [18] so that points $\{\sigma\}$ are closed in \widehat{N} because N is $*$ -regular.

Poguntke proved in Theorem 7 of [26] that if E is a simple $L^1(G)$ -module and N is a connected, coabelian, nilpotent subgroup of G , then there exists a unique orbit $G \cdot \sigma \subset \widehat{N}$ such that $\text{Ann}_{L^1(N)}(E) = k(G \cdot \sigma)$. More generally, Ludwig and Molitor-Braun showed in [21] that if T is a topologically irreducible, bounded representation of $L^1(G)$, then $\ker_{L^1(N)} T = k(G \cdot \sigma)$ for some $\sigma \in \widehat{N}$.

We need the following well-known facts about simple modules and minimal hermitian idempotents. In the following irreducible means topologically irreducible.

Lemma 4.3. *Let \mathcal{B} be Banach $*$ -algebra and π an irreducible $*$ -representation of \mathcal{B} in*

a Hilbert space \mathfrak{H} .

- (i) Let $\xi \in \mathfrak{H}$ be non-zero. Then the subspace $\pi(\mathcal{B})\xi$ is non-zero and dense in \mathfrak{H} . If I is an ideal of \mathcal{B} such that $I \not\subset \ker \pi$, then $\pi(I)\xi$ is also non-zero and dense.
- (ii) Suppose that the ideal I of all $f \in \mathcal{B}$ such that $\pi(f)$ has finite rank is non-zero. Then the $\pi(\mathcal{B})$ -invariant subspace $E = \pi(I)\mathfrak{H}$ generated by $\{\pi(f)\eta : f \in I, \eta \in \mathfrak{H}\}$ is a simple \mathcal{B} -module such that $\text{Ann}_{\mathcal{B}}(E) = \ker_{\mathcal{B}} \pi$.

Proof. Part (i) is obvious. The proof of (ii) follows Dixmier's proof of Théorème 2 in [7]. Let $\xi \in E$ be non-zero. For every $f \in I$ the subspace $\pi(f)\pi(\mathcal{B})\xi$ is dense in $\pi(f)\mathfrak{H}$ so that $\pi(f)\pi(\mathcal{B})\xi = \pi(f)\mathfrak{H}$ because $\pi(f)\mathfrak{H}$ is finite-dimensional. This proves $\pi(f)\mathfrak{H} \subset \pi(I)\xi$ for every $f \in I$. Thus $E = \pi(I)\mathfrak{H} = \pi(I)\xi$. The rest is obvious. \square

A hermitian idempotent $q \in \mathcal{B}$ satisfies $q^2 = q = q^*$. We say that q is minimal in \mathcal{B} if it is non-zero and if $q\mathcal{B}q = \mathbb{C}q$.

Lemma 4.4. *Let \mathcal{B} be Banach $*$ -algebra.*

- (i) Let π be a faithful irreducible $*$ -representation of \mathcal{B} in a Hilbert space \mathfrak{H} . Then $q \in \mathcal{B}$ is a minimal hermitian idempotent if and only if $\pi(q)$ is a one-dimensional orthogonal projection.
- (ii) Assume that there exist minimal hermitian idempotents in \mathcal{B} . If π, ρ are faithful irreducible $*$ -representations of \mathcal{B} , then π and ρ are unitarily equivalent.

Proof. Clearly q is a hermitian idempotent if and only if $\pi(q)$ is an orthogonal projection because π is faithful. If $\pi(q)\mathfrak{H}$ is a one-dimensional, then $\pi(\mathbb{C}q) = \mathbb{C}\pi(q) = \pi(q)\pi(\mathcal{B})\pi(q) = \pi(q\mathcal{B}q)$ and thus $q\mathcal{B}q = \mathbb{C}q$ because π is faithful. For the converse assume $q\mathcal{B}q = \mathbb{C}q$. Since $\pi(q\mathcal{B}q)$ and hence $\pi(q)$ acts irreducibly on $\pi(q)\mathfrak{H}$, it follows that this subspace is one-dimensional.

Now we prove (ii). Let $q \in \mathcal{B}$ be a minimal hermitian idempotent and π, ρ faithful irreducible $*$ -representations in Hilbert spaces \mathfrak{H}_π and \mathfrak{H}_ρ . Since $\pi(q)$ and $\rho(q)$ are one-dimensional orthogonal projections by (i), there exist unit vectors $\xi \in \mathfrak{H}_\pi$ and $\eta \in \mathfrak{H}_\rho$ such that $\pi(q) = \langle -, \xi \rangle \xi$ and $\rho(q) = \langle -, \eta \rangle \eta$. Let us consider the positive linear functionals f_π, f_ρ on \mathcal{B} given by $f_\pi(a) = \langle \pi(a)\xi, \xi \rangle$ and $f_\rho(a) = \langle \rho(a)\eta, \eta \rangle$. Since $q\mathcal{B}q$ is one-dimensional and $f_\pi(q) = 1 = f_\rho(q)$, it follows $f_\pi(a) = f_\pi(qaq) = f_\rho(qaq) = f_\rho(a)$ for all $a \in \mathcal{B}$, i.e., the positive linear forms of the cyclic representations π and ρ coincide. Now Proposition 2.4.1 of [8] shows that π and ρ are unitarily equivalent. \square

Poguntke proved in [25] that for exponential G and $\pi \in \widehat{G}$ there exists some $q \in L^1(G)$ such that $\pi(q)$ is a one-dimensional orthogonal projection. Note that the canonical image of q in $L^1(G)/\ker_{L^1(G)} \pi$ is a minimal hermitian idempotent. Part (ii) of Lemma 4.4 shows us that $\ker_{L^1(G)} \pi = \ker_{L^1(G)} \rho$ for $\pi, \rho \in \widehat{G}$ implies that π and ρ are unitarily equivalent. In particular G is a type I group. Furthermore the natural map $\Psi : \text{Prim } C^*(G) \longrightarrow \text{Prim}_* L^1(G)$ is injective, which is necessary for G to be primitive

$*$ -regular by Lemma 1.3.

If E is a simple \mathcal{B} -module, then there exists a complete norm on E such that $|a \cdot \xi| \leq |a| |\xi|$ for $a \in \mathcal{B}$ and $\xi \in E$: Recall that E is algebraically isomorphic to \mathcal{B}/L for some maximal modular left ideal L which is closed in the Banach algebra \mathcal{B} . The quotient norm of $E \cong \mathcal{B}/L$ has the desired property. In particular we see that primitive ideals $P = \text{Ann}_{\mathcal{B}}(E)$ are closed. Furthermore primitive ideals are prime. Hence the set $\text{Prim } \mathcal{B}$ of all primitive ideals of \mathcal{B} can be endowed with the Jacobson (hull-kernel) topology.

In the sequel we work with hermitian idempotents in the adjoint algebra \mathcal{B}^b of \mathcal{B} , compare [16], which is also known as the multiplier or double centralizer algebra of \mathcal{B} .

Proposition 4.5. *Let \mathcal{B} be a Banach $*$ -algebra and $q \in \mathcal{B}^b$ a hermitian idempotent.*

1. $q\mathcal{B}q$ is a closed $*$ -subalgebra of \mathcal{B} .
2. If E is a simple \mathcal{B} -module, then there exists a unique (simple) \mathcal{B}^b -module structure on E such that $M \cdot (a \cdot \xi) = (Ma) \cdot \xi$ for all $M \in \mathcal{B}^b$, $a \in \mathcal{B}$, and $\xi \in E$.
3. If E is a simple \mathcal{B} -module such that $q \cdot E \neq 0$, then $q \cdot E$ is a simple $q\mathcal{B}q$ -module with annihilator $\text{Ann}_{q\mathcal{B}q}(q \cdot E) = q\mathcal{B}q \cap \text{Ann}_{\mathcal{B}}(E) = q \text{Ann}_{\mathcal{B}}(E)q$.
4. The assignment $[E] \mapsto [q \cdot E]$ gives a bijection from the set of isomorphism classes of simple \mathcal{B} -modules E such that $q \cdot E \neq 0$ onto the set of isomorphism classes of simple $q\mathcal{B}q$ -modules.
5. Further $P \mapsto q\mathcal{B}q \cap P$ is a homeomorphism from the open subset $\text{Prim } \mathcal{B} \setminus h(\mathcal{B}q\mathcal{B})$ onto $\text{Prim}(q\mathcal{B}q)$ w. r. t. the Jacobson topology.

Proof. A proof of parts 1. to 4. of this proposition can also be found in [26].

1. Clearly $q\mathcal{B}q$ is a $*$ -subalgebra of \mathcal{B} because q is hermitian. The map $a \mapsto qaq$ is a continuous, linear projection. Its image $q\mathcal{B}q$ is closed.
2. Recall that \mathcal{B} is an ideal of \mathcal{B}^b . Let E be a simple \mathcal{B} -module. If $a \cdot \xi = 0$, then $\mathcal{B} \cdot (Ma) \cdot \xi = (\mathcal{B}M) \cdot (a \cdot \xi) = 0$ which implies $(Ma) \cdot \xi = 0$. Thus $M \cdot (a \cdot \xi) = (Ma) \cdot \xi$ defines a \mathcal{B}^b -module structure on E . The rest is obvious.
3. Let E be a simple \mathcal{B} -module. Clearly $q \cdot E$ is a $q\mathcal{B}q$ -module. If $0 \neq \xi \in q \cdot E$, then $(q\mathcal{B}q) \cdot \xi = q\mathcal{B} \cdot \xi = q \cdot E$. Thus $q \cdot E$ is simple. The equality for its annihilator is clear.
4. Since any simple \mathcal{B} -module is isomorphic to one of the form \mathcal{B}/L , L a maximal left ideal of \mathcal{B} , the isomorphism classes of simple \mathcal{B} -modules form a set. Note that any \mathcal{B} -linear map is also \mathcal{B}^b -linear.

The map $\alpha([E]) = [q \cdot E]$ is well-defined because any \mathcal{B} -linear isomorphism φ

from E_1 onto E_2 restricts to a $q\mathcal{B}q$ -linear isomorphism φ' from $q \cdot E_1$ onto $q \cdot E_2$. Further α is injective because any $q\mathcal{B}q$ -linear isomorphism $\varphi' : q \cdot E_1 \rightarrow q \cdot E_2$ extends to a \mathcal{B} -linear isomorphism $\varphi : E_1 \rightarrow E_2$: To see this, choose a non-zero $\xi \in q \cdot E$ and define $\varphi(a \cdot \xi) = a \cdot \varphi'(\xi)$. Finally, it remains to verify that α is surjective: Let E' be a simple $q\mathcal{B}q$ -module. Since $q\mathcal{B}^b q \subset (q\mathcal{B}q)^b$, we can define $E_0 = \mathcal{B} \otimes_{q\mathcal{B}^b q} E' = \mathcal{B}q \otimes_{q\mathcal{B}^b q} E'$. Observe that $q \cdot E_0 = q \otimes_{q\mathcal{B}^b q} E' \cong E'$. By Zorn's Lemma there exists a maximal \mathcal{B} -invariant subspace U of E_0 such that $U \cap q \cdot E_0 = \{0\}$. Put $E = E_0/U$. Clearly $q \cdot E \cong E'$. We claim that E is simple: If $\eta \notin U$, then the \mathcal{B} -invariant subspace $\tilde{U} = \mathcal{B} \cdot \eta + U$ satisfies $\tilde{U} \cap q \cdot E_0 \neq \{0\}$ and hence $q \cdot \tilde{U} \neq 0$. This implies $q\mathcal{B}q \cdot \tilde{U} = q \otimes_{q\mathcal{B}^b q} E'$ and $\mathcal{B}q\mathcal{B}q \cdot \tilde{U} = \mathcal{B} \otimes_{q\mathcal{B}^b q} E' = E_0$.

5. Part 3. implies $\beta(P) = q\mathcal{B}q \cap P \in \text{Prim}(q\mathcal{B}q)$ for all $P \in \text{Prim}(\mathcal{B}) \setminus h(\mathcal{B}q\mathcal{B})$. It follows from 4. that any simple $q\mathcal{B}q$ -module is isomorphic to one of the form $q \cdot E$, E a simple \mathcal{B} -module. Hence β is surjective. We will resort to the following preliminary remark: If $P \in \text{Prim}(q\mathcal{B}q)$ and I is an ideal of \mathcal{B} , then $I \subset P$ if and only if $qIq \subset qPq$. The only-if-part is obvious. Suppose $I \not\subset P$. Choose a simple \mathcal{B} -module E such that $q \cdot E \neq 0$ and $P = \text{Ann}_{\mathcal{B}}(E)$. If $a \in I$ and $a \notin P$, then $q\mathcal{B}a\mathcal{B}q \subset qIq$ and $q\mathcal{B}a\mathcal{B}q \cdot E = q \cdot E \neq 0$ which proves $qIq \not\subset qPq$. In particular the preceding remark shows that β is injective. Furthermore it is easy to see that β is continuous: If $A' \subset \text{Prim}(q\mathcal{B}q)$ is closed and $P \in \beta^{-1}(A')^-$, then $P \supset \cap \{Q : qQq \in A'\}$ and hence $qPq \supset Q'$ for all $Q' \in A$. This shows $qPq \in \overline{A'} = A'$ and thus $P \in \beta^{-1}(A')$. Finally we prove that β is a closed map: Suppose that A is a closed subset and P an element of $\text{Prim} \mathcal{B} \setminus h(\mathcal{B}q\mathcal{B})$ such that $qPq \in \beta(A)^-$ which means $qPq \supset \cap \{qQq : Q \in A\} \supset qk(A)q$. Now the preliminary remark implies $P \supset k(A)$ and thus $P \in \overline{A} = A$ and $\beta(P) \in \beta(A)$. This finishes our proof. □

Subalgebras of the form $q\mathcal{B}q$ for hermitian idempotents $q \in \mathcal{B}^b$ are called corners.

The representation theory of exponential Lie groups is dominated by the fact that certain subquotients $q * (L^1(G)/I) * q$ of the group algebra turn out to be isomorphic to (twisted) weighted convolution algebras on abelian groups.

In this context the smooth terminology of twisted covariance algebras (L^1 -version) is profitable, compare [13], [30]. By definition a twisted covariance system (G, \mathcal{A}, τ) consists of (1) a locally compact group G acting strongly continuously on a Banach $*$ -algebra \mathcal{A} by isometric $*$ -isomorphisms and (2) a twist τ defined on a closed normal subgroup H of G (i.e. a strongly continuous group homomorphism of H into the group of unitaries of the adjoint algebra \mathcal{A}^b of \mathcal{A}) such that $\tau(h^x) = \tau(h)^x$ and $a^h = \tau(h)^* a \tau(h)$ for all $x \in G$, $h \in H$, and $a \in \mathcal{A}$. Let $\mathcal{C}_0(G, \mathcal{A}, \tau)$ denote the space of all continuous functions $f : G \rightarrow \mathcal{A}$ such that $f(xh) = \tau(h)^* f(x)$ for all $x \in G$, $h \in H$ and such that f has compact support modulo H . The closure $L^1(G, \mathcal{A}, \tau)$ of $\mathcal{C}_0(G, \mathcal{A}, \tau)$ with respect to the norm $\|f\|_1 = \int_{G/H} |f(x)| dx$ is a Banach $*$ -algebra with convolution and

involution given by

$$(f * g)(x) = \int_{G/H} f(xy)y^{-1}g(y^{-1}) \, d\dot{y} \quad , \quad f^*(x) = \Delta_{G/H}(x^{-1}) (f(x^{-1})^*)^x .$$

A covariance pair (π, γ) is a unitary representation π of G and a $*$ -representation γ of \mathcal{A} in the same Hilbert space \mathfrak{H} such that $\gamma(a^x) = \pi(x)^*\gamma(a)\pi(x)$ and $\gamma(\tau(h)) = \pi(h)$. It is well-known that covariance pairs (π, γ) correspond to $*$ -representations of the twisted covariance algebra $L^1(G, \mathcal{A}, \tau)$.

Definition 4.6. Let (G, \mathcal{A}, τ) be a twisted covariance system. A family $\{\mathcal{A}_x : x \in G\}$ of closed subspaces of \mathcal{A} is said to be compatible with (G, \mathcal{A}, τ) if $\tau(h)^*\mathcal{A}_x = \mathcal{A}_{xh}$, $(\mathcal{A}_{xy})^{y^{-1}}\mathcal{A}_{y^{-1}} \subset \mathcal{A}_x$, and $((\mathcal{A}_{x^{-1}})^*)^x = \mathcal{A}_x$ for all $x, y \in G$ and $h \in H$.

If $\{\mathcal{A}_x : x \in G\}$ is compatible, then $\mathcal{C}_0(G, \mathcal{A}_x, \tau) = \{f \in \mathcal{C}_0(G, \mathcal{A}, \tau) : f(x) \in \mathcal{A}_x\}$ defines a subalgebra of $\mathcal{C}_0(G, \mathcal{A}, \tau)$. This bears a meaning only if the \mathcal{A}_x are chosen continuously so that $\mathcal{C}_0(G, \mathcal{A}_x, \tau)$ and hence its closure $L^1(G, \mathcal{A}_x, \tau)$ are non-zero. One might think of $\{\mathcal{A}_x : x \in G\}$ as a 'bundle' over G and ask for trivializations.

Definition 4.7. Let (G, \mathcal{A}, τ) be a twisted covariance system and $\{\mathcal{A}_x : x \in G\}$ a compatible family of one-dimensional subspaces of \mathcal{A} . We say that a continuous function $v : G \rightarrow \mathcal{A}$ is a trivialization for $\{\mathcal{A}_x : x \in G\}$ if $v(x) \in \mathcal{A}_x$, $|v(x)| \geq 1$, $v(xh) = \tau(h)^*v(x)$, $v(xy)y^{-1}v(y^{-1}) = v(x)$, and $(v(x^{-1})^*)^x = v(x)$ for $x, y \in G$, $h \in H$.

Proposition 4.8. Let (G, \mathcal{A}, τ) be a twisted covariance system. If v is a trivialization for the compatible family $\{\mathcal{A}_x : x \in G\}$ of one-dimensional subspaces of \mathcal{A} , then the subalgebra $L^1(G, \mathcal{A}_x, \tau)$ is isomorphic to the Beurling algebra $L^1(G/H, w)$ given by the symmetric weight function $w(\dot{x}) = |v(x)|$.

Proof. One checks easily that $\Phi(b)(x) = b(\dot{x})v(x)$ defines an isometric isomorphism from $L^1(G/H, w)$ onto $L^1(G, \mathcal{A}_x, \tau)$. \square

Let $q \in \mathcal{A} \subset L^1(G, \mathcal{A}, \tau)^b$ be a hermitian idempotent. Since $(q * f * q)(x) = q^x f(x) q$ for all $f \in L^1(G, \mathcal{A}, \tau)$, it follows $q * L^1(G, \mathcal{A}, \tau) * q = L^1(G, q^x \mathcal{A} q, \tau)$. In Theorem 4.9 we treat the case where q is minimal and the $q^x \mathcal{A} q$ are one-dimensional.

The following theorem is due to Poguntke, see part (4) and (5) of the proof of the main theorem in [25]. The idea goes back to Theorem 5 of Leptin and Poguntke in [20].

Theorem 4.9. Let (π, γ) be a covariance pair of the twisted covariance system (G, \mathcal{A}, τ) such that γ is irreducible and faithful. Suppose that there exists a minimal hermitian idempotent $q \in \mathcal{A}$. Then the corner $q * L^1(G, \mathcal{A}, \tau) * q = L^1(G, q^x \mathcal{A} q, \tau)$ is isometrically isomorphic to a weighted Beurling algebra $L^1(G/H, w)$ where w is a symmetric weight function on G/H .

Proof. By Lemma 4.4.(i) there exists a unit vector $\lambda \in \mathfrak{H}$ such that $\gamma(q)\xi = \langle \xi, \lambda \rangle \lambda$. Now $\gamma(q^x a q) = \pi(x)^*\gamma(q)\pi(x)\gamma(a)\gamma(q)$ implies

$$\gamma(q^x a q)\xi = \langle \pi(x)\gamma(a)\lambda, \lambda \rangle \langle \xi, \lambda \rangle \pi(x)^{-1}\lambda .$$

For every $x \in G$ there exists some $a \in \mathcal{A}$ such that $\langle \pi(x)\gamma(a)\lambda, \lambda \rangle$ is non-zero because γ is irreducible. This shows $\gamma(q^x \mathcal{A} q) = \mathbb{C} \pi(x)^{-1} \gamma(q)$ so that $q^x \mathcal{A} q$ is one-dimensional. There is a unique element $v(x) \in \mathcal{A}$ such that $\gamma(v(x))\xi = \langle \xi, \lambda \rangle \pi(x)^{-1} \lambda$. Clearly $v : G \rightarrow \mathcal{A}$ is continuous and $|v(x)| \geq |\gamma(v(x))| = 1$. Further one computes

$$\begin{aligned}\gamma(v(xh)) &= \pi(h)^* \gamma(v(x)) = \tau(h)^* \gamma(v(x)) \\ \gamma(v(x)) &= \gamma\left(v(xy)^{y^{-1}} v(y^{-1})\right) \\ \gamma(v(x)) &= \gamma((v(x^{-1}))^*)^x\end{aligned}$$

which proves that v is a trivialization for $\{q^x \mathcal{A} q : x \in G\}$ because γ is faithful. Now Proposition 4.8 gives the desired result. \square

Our aim is to apply Theorem 4.9 to certain quotients of group algebras: Let H be a closed normal subgroup of G . It is known that $L^1(G)$ is isomorphic to the twisted covariance algebra $L^1(G, L^1(H), \tau)$ with G -action $a^x(h) = \delta_H(x^{-1}) a(hx^{-1})$ and twist $\tau(k)a(h) = a(k^{-1}h)$, compare the corollary to Proposition 1 in [13]. Suppose that γ is an irreducible representation of H . The crucial assumption in Theorem 4.9 is that γ can be completed to a covariance pair (π, γ) of $(G, L^1(H), \tau)$, or equivalently, that it can be extended to a representation π of G , which is only possible if $x \cdot \gamma$ is unitarily equivalent to γ for all $x \in G$. If such a π exists, then the ideal $I' = \ker_{L^1(H)} \gamma$ is G - and τ -invariant so that the covariance algebra $L^1(G, L^1(H)/I', \tau)$ with induced G -action and twist is well-defined. This algebra is isomorphic to the quotient $L^1(G)/I$ where $I = \text{ind}_H^G(I')$. To apply Theorem 4.9 it remains to find minimal hermitian idempotents in $L^1(H)/I'$.

The existence of an extension π of γ is guaranteed under the assumptions of

Proposition 4.10. *Let G be an exponential Lie group, $f \in \mathfrak{g}^*$, and $\pi = \mathcal{K}(f) \in \widehat{G}$. Suppose that \mathfrak{h} is an ideal of \mathfrak{g} such that $\mathfrak{g} = \mathfrak{g}_f + \mathfrak{h}$. Let $f_0 = f|_{\mathfrak{h}}$ and $\gamma = \mathcal{K}(f_0)$. Then $\pi|_H$ is unitarily equivalent to γ . This means that π yields an extension of γ .*

Proof. Recall that there exists a \mathfrak{g}_f -invariant Pukanszky polarization $\mathfrak{p}_0 \subset \mathfrak{h}$ at $f_0 \in \mathfrak{h}^*$ because \mathfrak{g} is exponential, see §4, Chapter I of [19] and Chapter 5 of [1]. We shall verify that $\mathfrak{p} = \mathfrak{g}_f + \mathfrak{p}_0 \subset \mathfrak{g}$ defines a Pukanszky polarization at $f \in \mathfrak{g}^*$: Clearly $[\mathfrak{p}, \mathfrak{p}] \subset [\mathfrak{g}_f, \mathfrak{g}_f] + [\mathfrak{g}_f, \mathfrak{p}_0] + [\mathfrak{p}_0, \mathfrak{p}_0] \subset \mathfrak{p} \cap \ker f$. Note that $\mathfrak{p}_0 = \mathfrak{p} \cap \mathfrak{h}$ and $\mathfrak{h}_{f_0} = \mathfrak{g}_f \cap \mathfrak{h}$. Using the canonical isomorphisms $\mathfrak{h}/\mathfrak{h}_{f_0} \cong \mathfrak{g}/\mathfrak{g}_f$ and $\mathfrak{p}_0/\mathfrak{h}_{f_0} \cong \mathfrak{p}/\mathfrak{g}_f$ we conclude that $\dim \mathfrak{g}/\mathfrak{g}_f = \frac{1}{2} \dim \mathfrak{p}/\mathfrak{g}_f$. It remains to prove that $\text{Ad}^*(P)f = f + \mathfrak{p}^\perp$. If $h \in \mathfrak{p}^\perp$, then $h_0 = h|_{\mathfrak{h}} \in \mathfrak{p}_0^\perp \subset \mathfrak{h}^*$. Since \mathfrak{p}_0 is a Pukanszky polarization at f_0 , there exists some $x \in P_0$ such that $\text{Ad}^*(x)f_0 = f_0 + h_0$. This implies $\text{Ad}^*(x)f = f + h$ because $\mathfrak{g} = \mathfrak{g}_f + \mathfrak{h}$. Thus $\text{Ad}^*(P)f = f + \mathfrak{p}^\perp$.

Fix a relatively G -invariant measure on G/P . There exists a unique relatively H -invariant measure on H/P_0 such that the canonical H -equivariant diffeomorphism $H/P_0 \rightarrow G/P$ is measure-preserving. The modular functions of these measures satisfy

$\Delta_{G,P} \mid H = \Delta_{H,P_0}$. Now it follows that $\varphi \mapsto \varphi \mid H$ defines a unitary isomorphism from $L^2(G, \chi_f)$ onto $L^2(H, \chi_{f_0})$ which intertwines $\pi \mid H$ and γ . This completes the proof. \square

The next theorem results from the achievements of Poguntke in [26] concerning the parametrization of simple $L^1(G)$ -modules. In [21] Ludwig and Molitor-Braun gave a simplified proof of Theorem 4.11 which in particular avoids projective representations. The decisive idea of Ludwig and Molitor-Braun may be recapitulated as follows: If H/N is chosen to be a vector space complement to M/N instead of K/N as in [26], then one ends up directly with a commutative subquotient.

Recall that any simple $L^1(G)$ -module can be regarded as an $L^1(G)^b$ -module. In particular, if N is a closed subgroup of G , then E becomes an $L^1(N)$ -module so that $\text{Ann}_{L^1(N)}(E)$ is defined.

Let (G, \mathcal{A}) be a covariance system. A G -invariant ideal J of \mathcal{A} is called G -prime if $J_1 J_2 \subset J$ for G -invariant ideals J_1, J_2 of \mathcal{A} implies $J_1 \subset J$ or $J_2 \subset J$. If N is a closed normal subgroup of G , then one has the covariance system $(G, L^1(N))$ with G -action $a^x(n) = \delta_N(x^{-1}) a(n^{x^{-1}})$.

Theorem 4.11. *Let N be a closed, connected, coabelian, nilpotent subgroup of the exponential Lie group G .*

1. *If E is a simple $L^1(G)$ -module, then $J = \text{Ann}_{L^1(N)}(E)$ is G -prime.*
2. *Conversely let J be a G -prime ideal of $L^1(N)$. Define $I = \text{ind}_N^G(J)$. The simple $L^1(G)$ -modules E such that $J \subset \text{Ann}_{L^1(N)}(E)$ are in a canonical bijection with the simple modules of $\mathcal{B} = L^1(G)/I$. Moreover, there exist hermitian idempotents $q \in \mathcal{B}^b$ such that the corner $q * \mathcal{B} * q$ is commutative and such that $q \cdot E \neq 0$ exactly for those simple \mathcal{B} -modules E with $J = \text{Ann}_{L^1(N)}(E)$.*

Proof.

1. Recall that $\lambda(x)f(y) = f(x^{-1}y)$ defines a group homomorphism from G into the unitary group of $L^1(G)^b$. Since $a^x \cdot \xi = \lambda(x^{-1}) \cdot (a \cdot (\lambda(x) \cdot \xi))$ for $a \in L^1(N)$, $x \in G$, and $\xi \in E$, it follows that J is G -invariant. Now let J_1, J_2 be G -invariant ideals of $L^1(N)$ such that $J_1 * J_2 \subset J = \text{Ann}_{L^1(N)}(E)$. Then

$$\begin{aligned} \text{ind}_N^G(J_1) * \text{ind}_N^G(J_2) &\subset (L^1(G) * J_1 * L^1(G) * J_2 * L^1(G))^- \\ &\subset (L^1(G) * J_1 * J_2 * L^1(G))^- \subset \text{Ann}_{L^1(G)}(E). \end{aligned}$$

The first inclusion is obvious and the second one results from Lemma 2.11. For the third one we use the fact that $\text{Ann}_{L^1(G)}(E)$ is closed. Since this ideal is prime, it follows $\text{ind}_N^G(J_k) \subset \text{Ann}_{L^1(G)}(E)$ for $k = 1$ or 2 . Finally we obtain $J_k \subset \text{Ann}_{L^1(N)}(E)$ because E is a simple $L^1(G)$ -module.

2. Let J be a G -prime ideal of $L^1(N)$ and $I = \text{ind}_N^G(J)$. In order to prove the first assertion of 2. it suffices to verify that $J \subset \text{Ann}_{L^1(N)}(E)$ if and only if $I \subset \text{Ann}_{L^1(G)}(E)$. The only-if part is obvious. Suppose that $I \subset \text{Ann}_{L^1(G)}(E)$. Then $L^1(G) \cdot (J \cdot E) \subset \text{ind}_N^G(J) \cdot E = I \cdot E = 0$ implies $J \cdot E = 0$ because E is simple. This means $J \subset \text{Ann}_{L^1(N)}(E)$.

Next we prove the existence of appropriate hermitian idempotents in the adjoint algebra of $\mathcal{B} = L^1(G)/I$: Generalizing a theorem of Poguntke in [26], Ludwig and Molitor-Braun proved in Theorem 1.1.6 of [21] that there exists a unique orbit $G \cdot \sigma$ in \widehat{N} such that $J = k(G \cdot \sigma)$. Since the Kirillov map of N is bijective, we can choose $l \in \mathfrak{n}^*$ such that $\mathcal{K}(l) = \sigma$ and $f \in \mathfrak{g}^*$ such that $f|_{\mathfrak{n}} = l$. We stress that the definition of the stabilizer $\mathfrak{m} = \mathfrak{g}_f + \mathfrak{n}$ depends only on the orbit $\text{Ad}^*(G)l$, i.e., on the G -prime ideal J . As usual M denotes the closed, connected subgroup of G with Lie algebra \mathfrak{m} . In addition we fix a closed, connected subgroup H of G containing N such that H/N is complementary to M/N in the vector space G/N . In particular $G = G_f H$.

The ideal $I' = \text{ind}_N^H(J)$ is invariant under the G -action $b^x(h) = \delta_H(x^{-1})b(h^{x^{-1}})$ and the twist $\tau(k)b(h) = b(k^{-1}h)$ in $L^1(H)$. Since $I = \text{ind}_H^G(I')$, the quotient $\mathcal{B} = L^1(G)/I$ can be identified with $L^1(G, L^1(H)/I', \dot{\tau})$.

Let $f_0 = f|_{\mathfrak{h}} \in \mathfrak{h}^*$ and $\gamma = \mathcal{K}(f_0) \in \widehat{H}$. Since $\mathfrak{g} = \mathfrak{g}_f + \mathfrak{h}$, Proposition 4.10 implies that $\pi = \mathcal{K}(f)$ furnishes an extension of γ . Note that $\mathfrak{h}_{f_0} = \mathfrak{g}_f \cap \mathfrak{h} \subset \mathfrak{n}$. This shows that $\text{Ad}^*(H)f_0$ is saturated over \mathfrak{n} , i.e., $\text{Ad}^*(H)f_0 = f_0 + \mathfrak{n}^\perp$. In particular $\ker_{L^1(H)} \gamma$ is invariant under multiplication by characters of H/N , and hence $\mathcal{C}_\infty(H/N)$ -invariant. Now Theorem 2.10 implies $\ker_{L^1(H)} \gamma = \text{ind}_N^H(J) = I'$ because $\ker_{L^1(N)} \gamma = k(G \cdot \sigma) = J$ by Lemma 4.1. We have shown that γ yields a faithful irreducible representation of $\mathcal{A} = L^1(H)/I'$ which admits an extension π .

Let us fix an arbitrary minimal hermitian idempotent $q \in \mathcal{A} = L^1(H)/I'$. Since H is an exponential Lie group, the existence such idempotents is guaranteed by Poguntke's results in [25]. Finally Theorem 4.9 shows that the corner $q * \mathcal{B} * q$ is commutative as it is isomorphic to a weighted Beurling algebra $L^1(G/H, w)$ on the commutative group G/H .

Let E be a simple $L^1(G)$ -module such that $J \subset \text{Ann}_{L^1(N)}(E)$. It remains to be shown that $q \cdot E \neq 0$ if and only if $J = \text{Ann}_{L^1(N)}(E)$. The subsequent proof of the if-part is from [26]. We begin with a preliminary remark: Let Q'_0 denote the ideal of all $b \in L^1(H)$ such that $\gamma(b)$ has finite rank. Clearly $I' \subset Q'_0$ and $q \in Q'_0 \setminus I'$. Let \mathfrak{H} denote the representation space of γ , and F the simple module associated to γ in the sense of Lemma 4.3. It is known that $\gamma(Q'_0)$ is equal to the algebra of all finite rank operators A of \mathfrak{H} such that $A(\mathfrak{H}) \subset F$ and $A^*(\mathfrak{H}) \subset F$, compare Théorème 2. of Dixmier in [7]. From this we deduce $(Q'_0/I') * b * (Q'_0/I') = Q'_0/I'$

for all $b \in Q'_0 \setminus I'$. In particular we see that either $Q'_0 \subset \text{Ann}_{L^1(H)}(E)$ or $b \cdot E \neq 0$ for all $b \in Q'_0 \setminus I'$.

Suppose that $J = \text{Ann}_{L^1(N)}(E)$. Since $\text{Ad}^*(H)f_0$ is saturated over \mathfrak{n} , it follows that $\gamma \otimes \alpha$ is unitarily equivalent to γ for all $\alpha \in (H/N)^\wedge$. Thus Q'_0 and hence its closure Q' are $(H/N)^\wedge$ -invariant. By Theorem 2.10 there exists an H -invariant ideal R of $L^1(N)$ such that $Q' = \text{ind}_N^H(R)$. Note that R is not contained in J . Thus $Q = \text{ind}_N^G(R) = \text{ind}_H^G(Q')$ is not contained in $\text{Ann}_{L^1(G)}(E)$ because E is simple. Consequently Q'_0 is not contained in $\text{Ann}_{L^1(H)}(E)$ so that $q \cdot E \neq 0$ by the preliminary remark.

In order to prove the only-if-part we suppose $q \cdot E \neq 0$. The preceding remark implies $Q'_0 \cap \text{Ann}_{L^1(H)}(E) \subset I'$. Now we conclude $Q' * \text{Ann}_{L^1(H)}(E) \subset I'$. Since I' is G -prime and $Q', \text{Ann}_{L^1(H)}(E)$ are G -invariant ideals, we get $\text{Ann}_{L^1(H)}(E) \subset I'$ because $Q' \not\subset I'$. Let $a \in \text{Ann}_{L^1(N)}(E)$. Since $L^1(H) * a * L^1(H)$ is contained in $I' = \ker_{L^1(H)} \gamma$, it follows $a \in \ker_{L^1(N)} \gamma = k(G \cdot \sigma) = J$ because γ is irreducible.

□

Let J a given G -prime ideal of $L^1(N)$. In combination with part 4. of Proposition 4.5 the preceding theorem shows that the equivalence classes of all simple $L^1(G)$ -modules E with annihilator $\text{Ann}_{L^1(N)}(E) = J$ are in a one-to-one correspondence with the characters of the commutative Beurling algebra $q * (L^1(G)/I) * q \cong L^1(G/H, w)$.

Here we are content with this rough description and deliberately renounce more delicate questions such as obtaining estimates for the weight w , which can be found in [26].

Corollary 4.12. *If E, F are simple $L^1(G)$ -modules with $\text{Ann}_{L^1(G)}(E) \subset \text{Ann}_{L^1(G)}(F)$ and $J = \text{Ann}_{L^1(N)}(E) = \text{Ann}_{L^1(N)}(F)$, then E and F are isomorphic.*

Proof. Note that E and F can be regarded as \mathcal{B} -modules where $\mathcal{B} = L^1(G)/\text{ind}_N^G(J)$. Let $q \in \mathcal{B}^b$ be a hermitian idempotent as in part 2. of Theorem 4.11. By definition $q \cdot E$ and $q \cdot F$ are non-zero. Hence they are simple modules over the commutative $*$ -algebra $q * \mathcal{B} * q$ with annihilators $\text{Ann}_{q * \mathcal{B} * q}(q \cdot E) \subset \text{Ann}_{q * \mathcal{B} * q}(q \cdot F)$, compare Proposition 4.5. It results from Schur's Lemma that $q \cdot E$ and $q \cdot F$ are one-dimensional, have the same annihilator, and are thus isomorphic. Finally Proposition 4.5.4. shows that E and F are isomorphic as \mathcal{B} -modules, and also as $L^1(G)$ -modules. □

Corollary 4.13. *If $\pi, \rho \in \widehat{G}$ are irreducible such that $\ker_{L^1(G)} \pi \subset \ker_{L^1(G)} \rho$ and $\ker_{L^1(N)} \pi = \ker_{L^1(N)} \rho$, then π and ρ are unitarily equivalent.*

Proof. Let E, F denote the simple $L^1(G)$ -modules associated to π, ρ respectively in the sense of Lemma 4.3.(ii). By definition $\text{Ann}_{L^1(G)}(E) \subset \text{Ann}_{L^1(G)}(F)$ and $J = \text{Ann}_{L^1(N)}(E) = \text{Ann}_{L^1(N)}(F)$. Thus E and F are isomorphic by Corollary 4.12. This means $\ker_{L^1(G)} \pi = \text{Ann}_{L^1(G)}(E) = \text{Ann}_{L^1(G)}(F) = \ker_{L^1(G)} \rho$. Finally π and ρ are unitarily equivalent by Lemma 4.4.(ii). □

These preparations make it easy to prove the main result of this section.

Proposition 4.14. *Let G be an exponential Lie group and N a closed, connected, coabelian, nilpotent subgroup. Let $\pi \in \widehat{G}$ and $G \cdot \sigma$ be the unique G -orbit in \widehat{N} such that $k(G \cdot \sigma) = \ker_{C^*(N)} \pi$. If $G \cdot \sigma$ is closed in \widehat{N} , then $\ker_{C^*(G)} \pi$ is $L^1(G)$ -determined.*

Proof. Let ρ be in \widehat{G} such that $\ker_{L^1(G)} \pi \subset \ker_{L^1(G)} \rho$. Restricting to N we obtain $\ker_{L^1(N)} \pi \subset \ker_{L^1(N)} \rho$. Since N is $*$ -regular as a connected nilpotent Lie group, it follows $k(G \cdot \sigma) = \ker_{C^*(N)} \pi \subset \ker_{C^*(N)} \rho$. This yields $\ker_{C^*(N)} \pi = \ker_{C^*(N)} \rho$ because the orbit $G \cdot \sigma$ is closed. Finally Corollary 4.13 implies that π and ρ are unitarily equivalent so that in particular $\ker_{C^*(G)} \pi = \ker_{C^*(G)} \rho$. \square

However, the preceding results are limited to the case when $G \cdot \sigma$ is closed in \widehat{N} .

Remark 4.15. Let N be a coabelian, nilpotent subgroup of G and $\pi \in \widehat{G}$ such that $G \cdot \sigma$ is not closed in \widehat{N} . To prove that $\ker_{C^*(G)} \pi$ is $L^1(G)$ -determined, one must show that $\ker_{C^*(G)} \pi \not\subset \ker_{C^*(G)} \rho$ implies $\ker_{L^1(G)} \pi \not\subset \ker_{L^1(G)} \rho$ for all $\rho \in \widehat{G}$. Note that $J = \ker_{L^1(N)} \pi$ is G -prime and define $I = \text{ind}_N^G(J)$. To avoid trivialities we can assume $\ker_{L^1(N)} \pi \subset \ker_{L^1(N)} \rho$ so that π and ρ factor to representations of $\mathcal{B} = L^1(G)/I$. In addition we suppose that $\ker_{L^1(N)} \pi \neq \ker_{L^1(N)} \rho$. Such representations ρ are likely to exist if $G \cdot \sigma$ is not closed. If $q \in \mathcal{B}^b$ is a hermitian idempotent as in Theorem 4.11, then $\rho(q) = 0$. This means that restriction to the subquotient $q\mathcal{B}q$ is not appropriate for proving $\ker_{L^1(G)} \pi \not\subset \ker_{L^1(G)} \rho$ in this case.

5 A strategy to prove primitive $*$ -regularity

Let G be an exponential Lie group and \mathfrak{n} a coabelian nilpotent ideal of its Lie algebra \mathfrak{g} . In order to prove that G is primitive $*$ -regular, one must show that $\ker_{C^*(G)} \pi$ is $L^1(G)$ -determined for all $\pi \in \widehat{G}$, i.e., according to Definition 1.1 one must prove that

$$\ker_{C^*(G)} \pi \not\subset \ker_{C^*(G)} \rho \quad \text{implies} \quad \ker_{L^1(G)} \pi \not\subset \ker_{L^1(G)} \rho$$

for all $\rho \in \widehat{G}$. Let $f, g \in \mathfrak{g}^*$ such that $\pi = \mathcal{K}(f)$ and $\rho = \mathcal{K}(g)$. Since the Kirillov map of G is a homeomorphism with respect to the Jacobson topology on the primitive ideal space $\text{Prim } C^*(G)$ and the quotient topology on the coadjoint orbit space $\mathfrak{g}^*/\text{Ad}^*(G)$, the relation for the C^* -kernels is equivalent to $\text{Ad}^*(G)g \not\subset (\text{Ad}^*(G)f)^-$. From the preceding subsections we extract the following observations:

1. Let \mathfrak{a} be a non-trivial ideal of \mathfrak{g} such that $f = 0$ on \mathfrak{a} . Let A be the connected subgroup of G with Lie algebra \mathfrak{a} . Since $\pi = 1$ on A , we can pass over to a representation $\dot{\pi}$ of the quotient $\dot{G} = G/A$. It follows from Lemma 1.4 that $\ker_{C^*(G)} \pi$ is $L^1(G)$ -determined if and only if $\ker_{C^*(\dot{G})} \dot{\pi}$ is $L^1(\dot{G})$ -determined. Often \dot{G} is known to be primitive $*$ -regular by induction. If this is the case for all proper quotients \dot{G} of G , then we can assume that f is in general position, i.e., $f \neq 0$ on all non-trivial ideals \mathfrak{a} of \mathfrak{g} .

2. If the stabilizer $\mathfrak{m} = \mathfrak{g}_f + \mathfrak{n}$ is nilpotent, then $\ker_{C^*(G)} \pi$ is $L^1(G)$ -determined by Propositions 2.12 and 2.14 because M is $*$ -regular.
3. If $\mathfrak{g} = \mathfrak{g}_{f'} + \mathfrak{n}$, then $\ker_{C^*(G)} \pi$ is $L^1(G)$ -determined by Proposition 4.14. Here and in the sequel f' denotes the restriction of f to \mathfrak{n} .
4. If $\text{Ad}^*(G)g'$ is not contained in the closure of $\text{Ad}^*(G)f'$, then it follows $\ker_{C^*(N)} \pi \not\subset \ker_{C^*(N)} \rho$ because the Kirillov map is a homeomorphism. Since N is $*$ -regular, we obtain $\ker_{L^1(N)} \pi \not\subset \ker_{L^1(N)} \rho$ and hence $\ker_{L^1(G)} \pi \not\subset \ker_{L^1(G)} \rho$.

Lemma 5.1. *If there exists a one-codimensional nilpotent ideal \mathfrak{n} of \mathfrak{g} , then G is primitive $*$ -regular.*

Proof. Let $f \in \mathfrak{g}^*$ be arbitrary and $\pi = \mathcal{K}(f)$. The assumption $\dim \mathfrak{g}/\mathfrak{n} = 1$ implies that either $\mathfrak{g} = \mathfrak{g}_{f'} + \mathfrak{n}$, or that $\mathfrak{m} = \mathfrak{g}_f + \mathfrak{n} = \mathfrak{n}$ is nilpotent. Clearly the preceding remarks show that $\ker_{C^*(G)} \pi$ is $L^1(G)$ -determined. \square

Definition 5.2. Let $f \in \mathfrak{g}^*$ be in general position. As before $r : \mathfrak{g}^* \rightarrow \mathfrak{n}^*$ is given by $r(g) = g' = g|_{\mathfrak{n}}$. We define Ω as the r -preimage of the closure of $\text{Ad}^*(G)f'$ in \mathfrak{n}^* . Note that Ω is a closed subset of \mathfrak{g}^* containing $\text{Ad}^*(G)f$ and that $g \in \Omega$ if and only if g' is in the closure of $\text{Ad}^*(G)f'$. We say that g is critical for the orbit $\text{Ad}^*(G)f$ if $g \in \Omega \setminus (\text{Ad}^*(G)f)^-$. By Proposition 4.14 we can even assume $\text{Ad}^*(G)g' \neq \text{Ad}^*(G)f'$.

In order to prove the primitive $*$ -regularity of G it thus suffices to verify the following two assertions:

1. Every proper quotient \dot{G} of G is primitive $*$ -regular.
2. If $f \in \mathfrak{g}^*$ is in general position such that the stabilizer $\mathfrak{m} = \mathfrak{g}_f + \mathfrak{n}$ is a proper, non-nilpotent ideal of \mathfrak{g} and if $g \in \mathfrak{g}^*$ is critical for the orbit $\text{Ad}^*(G)f$, then it follows

$$\ker_{L^1(G)} \pi \not\subset \ker_{L^1(G)} \rho.$$

Let d_1, \dots, d_m be a coexponential basis for \mathfrak{m} in \mathfrak{g} . We obtain a diffeomorphism from \mathbb{R}^m onto G/M by composing the smooth map $E(s) = \exp(s_1 d_1) \cdots \exp(s_m d_m)$ with the quotient map $G \rightarrow G/M$. Define $\tilde{f} = f|_{\mathfrak{m}}$, $\tilde{f}_s = \text{Ad}^*(E(s))\tilde{f}$ in \mathfrak{m}^* , and $\tilde{\pi}_s = \mathcal{K}(\tilde{f}_s)$ in \widehat{M} .

Two properties of π and their counterpart in the Kirillov picture are worth mentioning. First $\pi|_M$ is reducible. By Lemma 4.1 we know that $\pi|_M$ is weakly equivalent to the subset $\{\tilde{\pi}_s : s \in \mathbb{R}^m\}$ of \widehat{M} . In the orbit picture $\text{Ad}^*(G)\tilde{f}$ decomposes into the disjoint union of the orbits $\{\text{Ad}^*(M)\tilde{f}_s : s \in \mathbb{R}^m\}$.

Secondly, $\ker_{C^*(G)} \pi$ is induced from M by Proposition 2.14. Hence $\ker_{C^*(G)} \pi \subset \ker_{C^*(G)} \rho$ is equivalent to the corresponding inclusion in $C^*(M)$. The same holds true in $L^1(M)$. In the Kirillov picture we have $\text{Ad}^*(G)f = \text{Ad}^*(G)f + \mathfrak{m}^\perp$ so that g is in $(\text{Ad}^*(G)f)^-$ if and only if \tilde{g} is in the closure of $\text{Ad}^*(G)\tilde{f}$.

In analogy to Definition 5.2 we define $\tilde{\Omega} \subset \mathfrak{m}^*$ and critical \tilde{g} for the orbit $\text{Ad}^*(G)\tilde{f}$ in \mathfrak{m}^* . We say that \tilde{f} is in general position if $f(\mathfrak{a}) \neq 0$ on any non-trivial ideal \mathfrak{a} of \mathfrak{g} such that $\mathfrak{a} \subset \mathfrak{m}$. Now it is easy to see that we can replace the second assertion by the following equivalent one:

3. Let \mathfrak{m} be a proper, non-nilpotent ideal of \mathfrak{g} such that $\mathfrak{m} \supset \mathfrak{n}$. If $\tilde{f} \in \mathfrak{m}^*$ is in general position such that $\mathfrak{m} = \mathfrak{m}_{\tilde{f}} + \mathfrak{n}$ and if $\tilde{g} \in \mathfrak{m}^*$ is critical for the orbit $\text{Ad}^*(G)\tilde{f}$, then the relation

$$(5.3) \quad \bigcap_{s \in \mathbb{R}^m} \ker_{L^1(M)} \tilde{\pi}_s \not\subset \ker_{L^1(M)} \tilde{\rho}$$

holds for the representations $\tilde{\pi}_s = \mathcal{K}(\tilde{f}_s)$ and $\tilde{\rho} = \mathcal{K}(\tilde{g})$.

In this situation producing functions $c \in L^1(M)$ such that $\pi_s(c) = 0$ for all s and $\rho(c) \neq 0$ turns out to be a great challenge.

Remark 5.4. The stabilizer $\mathfrak{m} = \mathfrak{g}_f + \mathfrak{n}$ has a remarkable algebraic property. Note that the ideal $[\mathfrak{m}, \mathfrak{z}\mathfrak{n}] = [\mathfrak{m}_f, \mathfrak{z}\mathfrak{n}]$ is contained in $\ker f$. If in addition f is in general position, then it follows $[\mathfrak{m}, \mathfrak{z}\mathfrak{n}] = 0$ so that $\mathfrak{z}\mathfrak{n} \subset \mathfrak{z}\mathfrak{m}$.

Lemma 5.5. *If \mathfrak{g} is an exponential Lie algebra such that $[\mathfrak{g}, \mathfrak{g}]$ is commutative, then G is $*$ -regular.*

Proof. Let $f \in \mathfrak{g}^*$ be arbitrary. If \mathfrak{a} denotes the largest ideal of \mathfrak{g} contained in $\ker f$, then \tilde{f} on $\tilde{\mathfrak{g}} = \mathfrak{g}/\mathfrak{a}$ is in general position. By Remark 5.4 we obtain $\tilde{\mathfrak{n}} = [\tilde{\mathfrak{g}}, \tilde{\mathfrak{g}}] = \mathfrak{z}\tilde{\mathfrak{n}} \subset \mathfrak{z}\tilde{\mathfrak{m}}$. Thus the quotient $\tilde{\mathfrak{m}}$ of $\mathfrak{m} = \mathfrak{g}_f + \mathfrak{n}$ is 2-step nilpotent so that \mathfrak{g} satisfies condition (R). Now Theorem 3.3 yields the assertion of this lemma. \square

6 A non- $*$ -regular example

Let \mathfrak{g} be an exponential Lie algebra of dimension ≤ 5 . In view of Lemma 5.1 and 5.5 we assume that the nilradical \mathfrak{n} (the maximal nilpotent ideal) of \mathfrak{g} is not commutative and of dimension ≤ 3 , i.e., $\mathfrak{n} = \langle e_1, e_2, e_3 \rangle$ is a 3-dimensional Heisenberg algebra. Further we suppose that $f \in \mathfrak{g}^*$ is general position (so that $f(e_3) \neq 0$) and that the stabilizer $\mathfrak{m} = \mathfrak{g}_f + \mathfrak{n}$ is a proper, non-nilpotent ideal. These assumptions imply that \mathfrak{g} has a basis d, e_0, \dots, e_3 satisfying the commutator relations $[e_1, e_2] = e_3$, $[e_0, e_1] = -e_1$, $[e_0, e_2] = e_2$, $[d, e_2] = e_2$, and $[d, e_3] = e_3$. The algebra \mathfrak{g} and the stabilizer $\mathfrak{m} = \langle e_0, \dots, e_3 \rangle$ are specified as $\mathfrak{g} = \mathfrak{b}_5$ and $\mathfrak{m} = \mathfrak{g}_{4,9}(0)$ in the (complete) list of all non-symmetric Lie algebras up to dimension 6 in [24], whereas symmetry is equivalent to $*$ -regularity by Theorem 10 of [26].

We work in coordinates of the second kind w. r. t. the above Malcev basis, given by the diffeomorphism $\Phi(t, x) = \exp(te_0) \exp(x_1 e_1) \exp(x_2 e_2 + x_3 e_3)$ from \mathbb{R}^4 onto M .

Denote $f|_{\mathfrak{m}}$ again by f and let $f_s = \text{Ad}^*(\exp(sd))f$. By choosing an appropriate representative on the orbit $\text{Ad}^*(G)f$ we can achieve $f(e_3) = 1$ and $f(e_1) = f(e_2) = 0$. Now we compute

$$\begin{aligned} \text{Ad}^*(\exp(sd)\Phi(t, x))f(e_0) &= f(e_0) - x_1x_2, \\ (e_1) &= e^tx_2, \\ (e_2) &= -e^{-s}e^{-t}x_1, \\ (e_3) &= e^{-s}. \end{aligned}$$

These formulas for the coadjoint representation suggest to define the $\text{Ad}^*(M)$ -invariant polynomial function

$$p = e_0e_3 - e_1e_2 - f(e_0)e_3$$

on \mathfrak{m}^* such that $p(h) = 0$ for all $h \in X = \text{Ad}^*(G)f = \cup\{\text{Ad}^*(M)f_s : s \in \mathbb{R}\}$. Here e_ν is interpreted as the linear function $e_\nu(h) = h(e_\nu)$ on \mathfrak{m}^* and $f(e_0)$ is a constant. Note that p is even $\text{Ad}^*(G)$ -semi-invariant. The closure of the orbit $X = \text{Ad}^*(G)f$ in \mathfrak{m}^* can be characterized by means of the $\text{Ad}^*(M)$ -invariant polynomial p .

Lemma 6.1. *Let $g \in \Omega \subset \mathfrak{m}^*$. Then $g \in \overline{X}$ if and only if $p(g) = 0$.*

Proof. The only-if-part is obvious because $p(h) = 0$ for all $h \in X$. Let us prove the opposite direction. Let $g \in \Omega$ such that $p(g) = 0$. We must distinguish four cases. First we assume $g(e_3) \neq 0$. Since $g \in \Omega$, it follows $g(e_3) > 0$. Without loss of generality we can assume $g(e_3) = 1 = f(e_3)$ and $g(e_1) = g(e_2) = 0$. Now $p(g) = 0$ implies $g(e_0) = f(e_0)$ so that $g \in X$. Next we consider the case $g(e_3) = g(e_2) = 0$ and $g(e_1) \neq 0$. If we define $s_n = n$, $x_{n1} = (g(e_0) - f(e_0))/g(e_1)$, and $x_{n2} = g(e_1)$, then it follows $f_n \rightarrow g$ for

$$f_n = \text{Ad}^*(\exp(s_nd)\Phi(0, x_n))f$$

in X so that $g \in \overline{X}$. The third case is $g(e_3) = g(e_1) = 0$ and $g(e_2) \neq 0$. If we set $s_n = n$, $x_{n1} = -e^n g(e_2)$, and $x_{n2} = -e^{-n}(f(e_0) - g(e_0))/g(e_2)$, then it follows $f_n \rightarrow g$. Finally we assume $g(e_\nu) = 0$ for $1 \leq \nu \leq 3$. In this case $s_n = n$, $x_{n1} = e^{n/2}$, and $x_{n2} = e^{-n/2}(f(e_0) - g(e_0))$ yields $f_n \rightarrow g$ so that $g \in \overline{X}$. \square

The preceding lemma implies that the set of critical linear functionals is given by $\Omega \setminus \overline{X} = \{g \in \mathfrak{m}^* : g(e_3) = 0 \text{ and } g(e_1)g(e_2) \neq 0\}$. Let us compute the relevant unitary representations: Using $\mathfrak{p} = \langle e_0, e_2, e_3 \rangle$ as a Pukanszky polarization at $f_s \in \mathfrak{m}^*$ for all $s \in \mathbb{R}$, one computes that $\pi_s = \mathcal{K}(f_s) = \text{ind}_P^M \chi_{f_s}$ in $L^2(\mathbb{R})$ is infinitesimally given by

$$\begin{aligned} d\pi_s(\dot{e}_0) &= f_0 + \xi D_\xi - i/2, \\ d\pi_s(\dot{e}_1) &= -D_\xi, \\ d\pi_s(\dot{e}_2) &= e^{-s}\xi, \\ d\pi_s(\dot{e}_3) &= e^{-s}. \end{aligned}$$

Here $\dot{e}_\nu = -ie_\nu$ is in the complexification $\mathfrak{m}_\mathbb{C}$ of \mathfrak{m} , $\xi \cdot -$ is the multiplication operator and $D_\xi = -i\partial_\xi$ is the differential operator in $L^2(\mathbb{R})$. We observe that these equations

bear a striking resemblance to the formulas for $\text{Ad}^*(\Phi(t, x))f(e_\nu)$: simply substitute $e^{-t}x_1$ by ξ and e^tx_2 by D_ξ . On the other hand, if $g \in \Omega \setminus \overline{X}$, then \mathfrak{n} is a Pukanszky polarization at $g \in \mathfrak{m}^*$ and $\rho = \mathcal{K}(g) = \text{ind}_N^M \chi_g$ in $L^2(\mathbb{R})$ is given by

$$\begin{aligned} d\rho(\dot{e}_0) &= -D_\xi, \\ d\rho(\dot{e}_1) &= e^\xi g_1, \\ d\rho(\dot{e}_2) &= e^{-\xi} g_2, \\ d\rho(\dot{e}_3) &= 0. \end{aligned}$$

Symmetrization gives a linear isomorphism from the symmetric algebra $\mathcal{S}(\mathfrak{m}_\mathbb{C}) = \mathcal{P}(\mathfrak{m}^*)$ onto the universal enveloping algebra $\mathcal{U}(\mathfrak{m}_\mathbb{C})$ of $\mathfrak{m}_\mathbb{C}$, which maps the subspace of $\text{Ad}(M)$ -invariant polynomials onto the center $Z(\mathfrak{m}_\mathbb{C})$ of $\mathcal{U}(\mathfrak{m}_\mathbb{C})$, compare Chapter 3.3 of [6]. Note that $p \in \mathcal{P}(\mathfrak{m}^*)^{\text{Ad}(M)}$ corresponds to

$$W = \dot{e}_3 \dot{e}_0 - \frac{1}{2}(\dot{e}_2 \dot{e}_1 + \dot{e}_1 \dot{e}_2) - f_0 \dot{e}_3 = \dot{e}_3 \dot{e}_0 - \dot{e}_2 \dot{e}_1 - (f_0 - \frac{i}{2}) \dot{e}_3$$

in $Z(\mathfrak{m}_\mathbb{C})$. One verifies easily that $d\tau(W) = p(h)$ holds for all $h \in \mathfrak{m}^*$ and $\tau = \mathcal{K}(h)$. For the Lie algebra \mathfrak{m} under consideration the symmetrization map coincides with the so-called Duflo isomorphism so that $d\tau(W) = p(h)$ can also be seen as a consequence of Théorème 2 of [9].

Furthermore we recall that if $\lambda(m)a(y) = a(m^{-1}y)$ denotes the left regular representation of M in $L^2(M)$, then

$$d\lambda(X)a(y) = \frac{d}{dt}\bigg|_{t=0} a(\exp(-tX)y)$$

defines a representation of \mathfrak{m} in $\mathcal{C}_0^\infty(M)$, which extends to $\mathcal{U}(\mathfrak{m}_\mathbb{C})$. Note that $\mathcal{U}(\mathfrak{m}_\mathbb{C})$ acts as an associative algebra of right invariant vector fields. Let us write $V * a = d\lambda(V)a$ for $V \in \mathcal{U}(\mathfrak{m}_\mathbb{C})$ and $a \in \mathcal{C}_0^\infty(M)$. It is known that $\tau(V * a) = d\tau(V)\tau(a)$ holds for all V, a and all unitary representations τ of M .

Lemma 6.2. *If $g \in \Omega \setminus \overline{X}$ and $\rho = \mathcal{K}(g)$, then $\bigcap_{s \in \mathbb{R}} \ker_{L^1(M)} \pi_s \not\subset \ker_{L^1(M)} \rho$. In particular G is primitive $*$ -regular.*

Proof. Since $\mathcal{C}_0^\infty(M)$ is dense in $L^1(M)$, there exists a function $a \in \mathcal{C}_0^\infty(M)$ such that $\rho(a) \neq 0$. Now $b = W * a$ satisfies $\pi_s(b) = d\pi_s(W)\pi_s(a) = p(f_s)\pi_s(a) = 0$ for all $s \in \mathbb{R}$ and $\rho(b) = d\rho(W)\rho(a) = p(g)\rho(a) \neq 0$ because $g \notin \overline{X}$. \square

A priori this result does not seem unlikely because the nature of X is essentially different from that of typical non- $*$ -regular subsets of $\mathfrak{m}^*/\text{Ad}^*(M)$. In the preceding lemma $X/\text{Ad}^*(M)$ is a graph over \mathfrak{zm}^* in the sense that the orbit $\text{Ad}^*(M)h$ is uniquely determined by $h|_{\mathfrak{zm}}$ for all $h \in X$. Whereas basic examples of non- $*$ -regular subsets X consist of linear functionals $h \in \mathfrak{m}^*$ over a common character $\zeta = h|_{\mathfrak{zm}}$ of the center such that the set of limit points of $X/\text{Ad}^*(M)$ in $\mathfrak{m}^*/\text{Ad}^*(M)$ is not empty.

Since $\mathfrak{g} = \mathfrak{b}_5$ is the only exponential Lie algebra in dimension ≤ 5 such that there exist $f \in \mathfrak{g}^*$ in general position with non-nilpotent, proper stabilizer and critical functionals $g \in \mathfrak{g}^*$ w. r. t. the orbit $\text{Ad}^*(G)f$, it follows from Lemma 6.2 that all exponential Lie groups up to dimension 5 are primitive $*$ -regular.

Note that in the particular case $\mathfrak{g} = \mathfrak{b}_5$ the relation $\cap_s \ker_{\mathcal{U}(\mathfrak{m}_{\mathbb{C}})} \pi_s \not\subset \ker_{\mathcal{U}(\mathfrak{m}_{\mathbb{C}})} \rho$ implies $\cap_s \ker_{L^1(M)} \pi_s \not\subset \ker_{L^1(M)} \rho$, but in general, as one might expect, the features of the universal enveloping algebra do not suffice for this purpose. However, we anticipate that $\text{Ad}(M)$ -invariant polynomials p corresponding to elements $W \in Z(\mathfrak{m}_{\mathbb{C}})$ will play an important role in further investigations of primitive $*$ -regularity.

Acknowledgment. The author would like to thank D. Poguntke for suggesting the possibility of proving Proposition 4.14 by means of the results in [26]. This article owes a lot to his valuable remarks and comments.

References

- [1] P. Bernat, N. Conze, M. Duflo, M. Lévy-Nahas, M. Raïs, P. Renouard, and M. Vergne. *Représentations des groupes de Lie résolubles*. Dunod, Paris, 1972. Monographies de la Société Mathématique de France, No. 4.
- [2] J. Boidol. $*$ -regularity of exponential Lie groups. *Invent. Math.*, 56(3):231–238, 1980.
- [3] J. Boidol. Connected groups with polynomially induced dual. *J. Reine Angew. Math.*, 331:32–46, 1982.
- [4] J. Boidol, H. Leptin, J. Schürman, and D. Vahle. Räume primitiver Ideale von Gruppenalgebren. *Math. Ann.*, 236(1):1–13, 1978.
- [5] Ian D. Brown. Dual topology of a nilpotent Lie group. *Ann. Sci. École Norm. Sup. (4)*, 6:407–411, 1973.
- [6] Lawrence J. Corwin and Frederick P. Greenleaf. *Representations of nilpotent Lie groups and their applications. Part I*, volume 18 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1990. Basic theory and examples.
- [7] Jacques Dixmier. Opérateurs de rang fini dans les représentations unitaires. *Inst. Hautes Études Sci. Publ. Math.*, (6):13–25, 1960.
- [8] Jacques Dixmier. *Les C^* -algèbres et leurs représentations*. Deuxième édition. Cahiers Scientifiques, Fasc. XXIX. Gauthier-Villars Éditeur, Paris, 1969.
- [9] Michel Duflo. Opérateurs différentiels bi-invariants sur un groupe de Lie. *Ann. Sci. École Norm. Sup. (4)*, 10(2):265–288, 1977.
- [10] Jacek Dziubanski, Jean Ludwig, and Carine Molitor-Braun. Functional calculus in weighted group algebras. *Rev. Mat. Complut.*, 17(2):321–357, 2004.
- [11] J. M. G. Fell. Weak containment and induced representations of groups. *Canad. J. Math.*, 14:237–268, 1962.

- [12] J. M. G. Fell. Weak containment and Kronecker products of group representations. *Pacific J. Math.*, 13:503–510, 1963.
- [13] Philip Green. The local structure of twisted covariance algebras. *Acta Math.*, 140(3-4):191–250, 1978.
- [14] Wilfried Hauenschild and Jean Ludwig. The injection and the projection theorem for spectral sets. *Monatsh. Math.*, 92(3):167–177, 1981.
- [15] J. W. Jenkins. Growth of connected locally compact groups. *J. Functional Analysis*, 12:113–127, 1973.
- [16] Horst Leptin. Verallgemeinerte L^1 -Algebren und projektive Darstellungen lokal kompakter Gruppen. I, II. *Invent. Math.* 3 (1967), 257–281; *ibid.*, 4:68–86, 1967.
- [17] Horst Leptin. Darstellungen verallgemeinerter L^1 -Algebren. *Invent. Math.*, 5:192–215, 1968.
- [18] Horst Leptin. Symmetrie in Banachschen Algebren. *Arch. Math. (Basel)*, 27(4):394–400, 1976.
- [19] Horst Leptin and Jean Ludwig. *Unitary representation theory of exponential Lie groups*, volume 18 of *de Gruyter Expositions in Mathematics*. Walter de Gruyter & Co., Berlin, 1994.
- [20] Horst Leptin and Detlev Poguntke. Symmetry and nonsymmetry for locally compact groups. *J. Funct. Anal.*, 33(2):119–134, 1979.
- [21] J. Ludwig and C. Molitor-Braun. Représentations irréductibles bornées des groupes de Lie exponentiels. *Canad. J. Math.*, 53(5):944–978, 2001.
- [22] Calvin C. Moore and Jonathan Rosenberg. Groups with T_1 primitive ideal spaces. *J. Functional Analysis*, 22(3):204–224, 1976.
- [23] Detlev Poguntke. Nilpotente Liesche Gruppen haben symmetrische Gruppen-algebren. *Math. Ann.*, 227(1):51–59, 1977.
- [24] Detlev Poguntke. Nichtsymmetrische sechsdimensionale Liesche Gruppen. *J. Reine Angew. Math.*, 306:154–176, 1979.
- [25] Detlev Poguntke. Operators of finite rank in unitary representations of exponential Lie groups. *Math. Ann.*, 259(3):371–383, 1982.
- [26] Detlev Poguntke. Algebraically irreducible representations of L^1 -algebras of exponential Lie groups. *Duke Math. J.*, 50(4):1077–1106, 1983.
- [27] L. Pukánszky. On the unitary representations of exponential groups. *J. Functional Analysis*, 2:73–113, 1968.
- [28] Iain Raeburn and Dana P. Williams. *Morita equivalence and continuous-trace C^* -algebras*, volume 60 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 1998.
- [29] Hans Reiter and Jan D. Stegeman. *Classical harmonic analysis and locally compact groups*, volume 22 of *London Mathematical Society Monographs. New Series*. The Clarendon Press Oxford University Press, New York, second edition, 2000.
- [30] Marc A. Rieffel. Induced representations of C^* -algebras. *Advances in Math.*, 13:176–257, 1974.

- [31] Marc A. Rieffel. Unitary representations of group extensions; an algebraic approach to the theory of Mackey and Blattner. In *Studies in analysis*, volume 4 of *Adv. in Math. Supl. Stud.*, pages 43–82. Academic Press, New York, 1979.

O. Ungermann
Fakultät für Mathematik
Universität Bielefeld
Postfach 100131
D-33501 Bielefeld
Germany
oungerma@math.uni-bielefeld.de